Math 155 Fall 2016
Exam 1

1. (15 Points) Suppose that the height of Graham's beanstalk at time t is given by

$$H(t) = H_0 e^{1.05t},$$

Where H_0 is the initial length (in cm), and t is measured in weeks.

(a) Find the time it takes for the beanstalk to double in height.

(b) If the beanstalk is 3 cm tall at time $t = 2$, how tall was the bean stalk at time $t = 0$? (That is, what is H_0?)

(c) Consider the discrete-time dynamical system with updating function

$$k_{t+1} = 1.35k_t$$

where k_t is the population of katydids at time t, measured in thousands. Find the (explicit) solution to the discrete-time dynamical system, given the initial condition $k_0 = 4$. Express your solution in terms of an exponential function in base e.

2. (15 Points) A baby polar bear splashes into her salt-water pool at Clayton's Wildlife Park, spilling 28 L of water. Her pool usually holds 6000 L of water. If one replaces the 28 L of water with salt water that has a concentration of 3 mol/L, the concentration of salt may change.

(a) Fill in the blank boxes below to model the situation above. Let s_t represent the concentration of salt in the pool after the bear has splashed t times, measured in mol/L. Remember that concentration is equal to the amount of salt (mol) divided by the volume (L).

Step	**Volume (L)**	**Total Salt (mol)**	**Salt Concentration (mol/L)**
H_2O in pool before bear jumps in	6000		s_t
Water lost	28		s_t
H_2O in pool after bear jumps in	5972		s_t
Water replaced	28		
H_2O in pool after replacing water			

(b) Write down the discrete-time dynamical system derived from the chart in part (a):

$$s_{t+1} =$$

(c) Suppose that Vanessa mixes 23 L of water with a salt concentration of C_1 moles/L with 41 L of water with a salt concentration of C_2 moles/L. Express the salt concentration of the resulting mixtures in terms of weighted average.

3. (14 Points)

(a) Write down a discrete-time dynamical system and an initial condition to describe the following situation: Each day, Jessica uses up 12% of the medicine her bloodstream. However, she takes enough medicine at the end of each day to increase the concentration of the medicine in her bloodstream by 2 milligrams per liter. She starts with a concentration of medicine in her bloodstream equal to 1 milligram per liter. (Let $M_t =$ the concentration of medicine on day *t*, in milligrams per liter.)

(b) Find all equilibria of the discrete-time dynamical system

$$x_{t+1} = \frac{px_t}{3x_t - 1},$$

where *p* is a parameter. For what range of values of *p* is there a positive equilibrium?

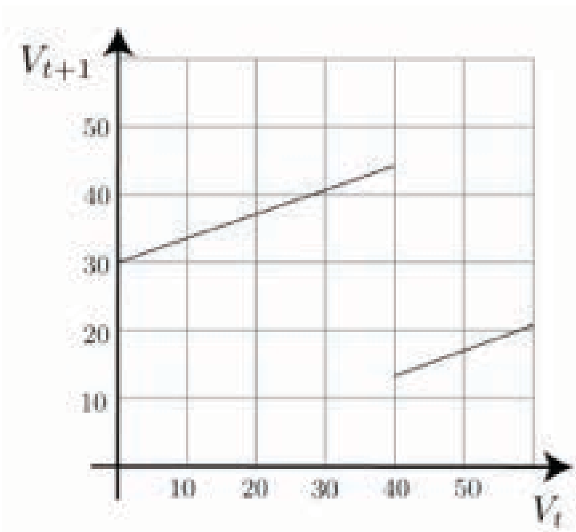
(c) Suppose that a newborn ram grows (in volume) by 42 cm^3/week. Given that 1 week = 7 days, 1 day = 24 hours, and 1 in = 2.54 cm, how fast is the newborn ram growing in in^3/hour? Write out conversation factors and show your work for credit.

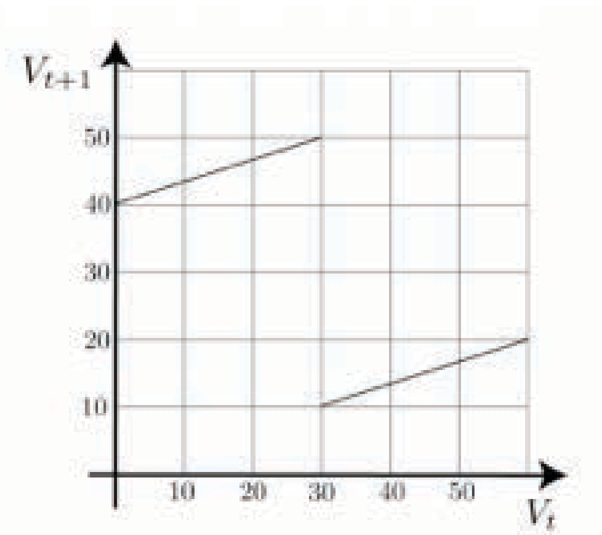
4. (14 Points) Let V_t represent the voltage at the AV node in the heart model

$$V_{t+1} = \begin{cases} e^{-\alpha\tau} + u, & \text{if } V_t \leq e^{\alpha\tau} V_C \\ e^{-\alpha\tau}, & \text{if } V_t > e^{\alpha\tau} V_C \end{cases}$$

(a) For each of the following two graphs of the updating function, cobweb starting from an initial value of $V_0 = 50$, and determine if the heart

i) is healthy, ii) has a 2:1 block, or iii) has the Wenckebach phenomenon.
Include arrows on your cobweb diagram.





(b) Now let $e^{-\alpha\tau} = \frac{1}{8}$, $u = 4$, and $V_C = 2$. Does the system have an equilibrium? Justify your answer algebraically (that is, without drawing a graph), and find the equilibrium if there is one.

5. (14 Points) Find the following limits, if they exist. Show all of your work, and justify your answer to receive full credit. If a limit does not exist, write "DNE," and explain why it does not exist.

i) $\lim\limits_{x \to -1} \dfrac{x^2 - 2x - 3}{x + 1}$

ii) $\lim\limits_{x \to 5} \dfrac{4}{(x-5)^{14}}$

iii) $\lim\limits_{x \to 2} f(x)$, where $f(x) = \begin{cases} 4x-5, & \text{if } x<2 \\ 4e^{x-2}, & \text{if } x \geq 2 \end{cases}$

iv) $\lim\limits_{\Delta t \to 0} \dfrac{-2(t + \Delta t)^2 + 2t^2}{\Delta t}$

6. (14 Points)

(a) We are interested in the density of a substance at a temperature of absolute zero (which is 0 Kelvin). However, we cannot measure the density directly at 0 Kelvin because it is impossible to reach absolute 0. Instead, we measure density for small values of the temperature.

i) What is $\lim\limits_{T \to 0^+} \dfrac{100}{2+T}$?

ii) How close to 0 Kelvin would the temperature have to be for the density to be within 1% of the limit?

(b) Consider the function

$$f(x) = \begin{cases} 2-x, & \text{if } x<2; \\ 0, & \text{if } x=2; \\ \ln(x-1), & \text{if } x>2; \end{cases}$$

Find the following limits, if they exist. If a limit does not exist, write "DNE," and explain why it does not exist.

$$\lim_{x \to 2^-} f(x)$$

$$\lim_{x \to 2^+} f(x)$$

$$\lim_{x \to 2} f(x)$$

Is this function continuous at x=2? Why or why not? Use the definition of continuity (not a phrase like "the graph can be drawn without lifting your pencil") to justify your answer.

7. (14 Points) Dean throws a ball up into the air from the top of a tower. Suppose that the height *h(t)* (in meters) of the ball as a function of time *t* (in seconds) is given by

$$h(t) = -5t^2 + 20t + 10.$$

(a) Find a formula for the slope of the secant line that passes through the points $(3, h(3))$ and $(3+\Delta t, h(3+\Delta t))$. Simplify your answer.

(b) Find the average rate of change in *h* between time $t = 3$ and $t = 3.5$.

(c) Find the instantaneous rate of change of *h* at $t = 3$ *using the limit definition of the derivative/instantaneous rate of change.* Is the height of the ball increasing or decreasing at time $t = 3$?

(d) Find the equation for the tangent line to the graph of $h(t)$ at the point (3, *h*(3)).