1. (14 pts)

(a) After 2 minutes on a treadmill, Niki has a heart rate is 84 beats per minute. After 4 minutes, her heart rate is 88 beats per minute.

i. Assuming Niki's heart rate h is a linear function of time t spent on the treadmill, give an equation for $h(t)$ in slope-intercept form.

Points: $(2,84)$, $(4,88)$

slope $m = \frac{88-84}{4-2} = \frac{4}{2} = 2$

$h(t) = 2(t-2)+84$

$\boxed{h(t) = 2t+80}$ in slope-intercept form.

ii. What will be Niki's heart rate after half an hour on the treadmill?

half-hour $\Rightarrow$ $t = 30$ minutes

$h(30) = 2(30)+80$

$\boxed{= 140 \frac{\text{beats}}{\text{min}}}$

iii. If Niki's heart rate is 120 beats per minute, how long has she been on the treadmill?

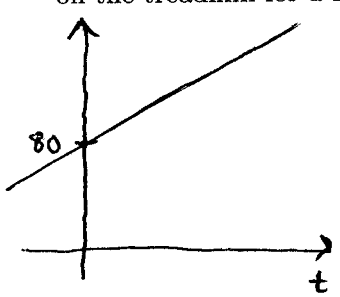
$120 = h(t) = 2t+80$

$120-80 = 2t$

$20 = t$

$\therefore$ She has been on the treadmill for 20 minutes.

iv. Graph the function $h(t)$ and describe what will happen to Niki if she decides to run on the treadmill for a few hours?



The function $h(t)$ increases as time t increases. The longer she runs, the faster the heart beats. If she runs too long, her heart will explode.

(b) A snowman is made of two spherical snowballs. The head has a radius of 1 foot and the body has a radius of 2.5 feet. Find the volume of snow used to make the snowman. Write your answer in cubic meters. You may use the following facts: 1 inch = 2.54 cm, 1 m =100 cm, and the volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.

One method:

$r_1 = 1\text{ ft} \times \frac{12\text{ in}}{1\text{ ft}} \times \frac{2.54\text{ cm}}{1\text{ in}} \times \frac{1\text{ m}}{100\text{ cm}} = 0.3048\text{ m}$

$r_2 = 2.5\text{ ft} \times \frac{12\text{ in}}{1\text{ ft}} \times \frac{2.54\text{ cm}}{1\text{ in}} \times \frac{1\text{ m}}{100\text{ cm}} = 0.762\text{ m}$

Total volume $= V_{small} + V_{big}$

$= \frac{4}{3}\pi r_1^3 + \frac{4}{3}\pi r_2^3$

$= \frac{4}{3}\pi(0.3048\text{ m})^3 + \frac{4}{3}\pi(0.762\text{ m})^3$

$= 0.1186\text{ m}^3 + 1.853\text{ m}^3$

$\boxed{= 1.97\text{ m}^3}$

2nd method:

Total volume $= V_{small} + V_{big}$

$= \frac{4}{3}\pi(1\text{ ft})^3 + \frac{4}{3}\pi(2.5\text{ ft})^3$

$= 4.189\text{ ft}^3 + \quad \text{ft}^3$

$= 69.639\text{ ft}^3$

$69.639\text{ ft}^3 = 69.639\text{ ft}^3 \cdot \frac{(12\text{ in})^3}{(1\text{ ft})^3} \frac{(2.54\text{ cm})^3}{(1\text{ in})^3} \frac{(1\text{ m})^3}{(100\text{ cm})^3}$

$\boxed{= 1.97\text{ m}^3}$

2. (14 points) The amount (in micrograms) of C^{14} left t years after death of an organism is given by

$$Q(t) = Q_0 e^{-0.000122t},$$

where Q_0 is the amount of C^{14} at the time of death.

(a) When will half of the original amount of C^{14} be left in the remains?

$$\frac{1}{2}Q_0 = Q_0 e^{-0.000122t_h}$$

$$\frac{1}{2} = e^{-0.000122t_h}$$

$$\ln(1/2) = -0.000122t_h;$$

$$t_h = \frac{\ln(1/2)}{-0.000122} \sim 5681.534$$

(b) If 0.015 micrograms are present after 10,000 years, how much C^{14} was in the organism at the time of death? (That is, find Q_0.)

$$Q(t) = Q_0 e^{-0.000122t}$$

$$0.015 = Q_0 e^{-0.000122(10,000)};$$

$$Q_0 = \frac{0.015}{e^{-0.000122(10,000)}} \cong 0.051$$

(c) Consider the discrete-time dynamical system with updating function

$$b_{t+1} = 0.6b_t,$$

where b_t is the population of bacteria at time t, measured in thousands. Find the solution $b(t)$ to the discrete-time dynamical system, given the inital condition $b_0 = 9$. Express your solution in terms of an exponential function in base e.

$$b_t = b_0 r^t$$

$$= 9(0.6)^t = 9e^{\ln(0.6)t}$$

$$\boxed{b_t = 9e^{\ln(0.6)t}}$$

3. (14 points) A colony of bacteria is composed of wild-type (b_t) and mutant-type (m_t) bacteria. Further, you observe that 25% mutants revert to wild-type and 10% of the wild-type mutate each day, and there is no reproduction.

   (a) Construct an updating function for the population b_{t+1} of wild type as a function of b_t and m_t.

$$b_{t+1} = (1-0.10)b_t + 0.25m_t$$

$$\boxed{b_{t+1} = 0.90\, b_t + 0.25 m_t}$$

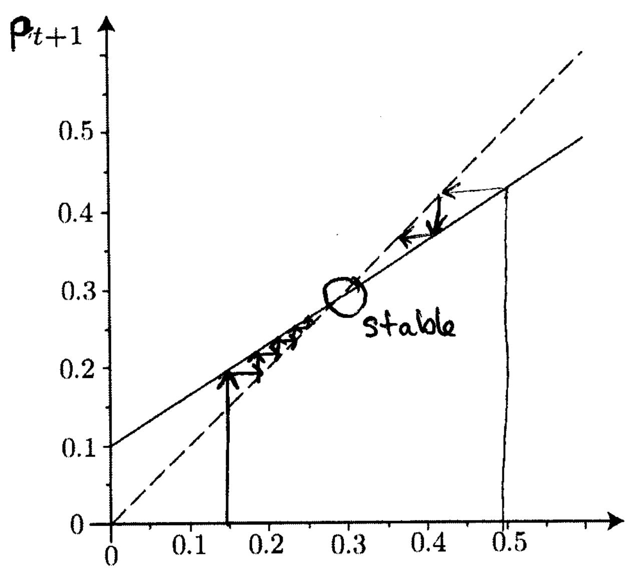
   (b) Let p_t be the fraction of mutants. The discrete-time dynamical system for this fraction is given by $p_{t+1} = 0.65p_t + 0.10$ (you do not need to derive this). What is the equilibrium fraction of mutants?

$$p^* = 0.65p^* + 0.10$$

$$0.35\,p^* = 0.10$$

$$p^* = \frac{0.10}{0.35} \sim 0.286$$

   (c) The updating function for the DTDS $p_{t+1} = 0.65p_t + 0.10$ is graphed below.
   i) Cobweb for five steps, starting at $p_0 = 0.15$. ii) Circle any equilibria and determine if each equilibrium is stable or unstable. iii) Will either the wild-type or mutant-type populations become extinct? If not, which population is larger in the long run? (Note: $p_t = \frac{m_t}{m_t+b_t}$.)



p_t

(iii) P_t tends to the equilibrium value 0.286. Since P_t is the fraction of mutants, and $0.286 < 1$, so the population of wild-type is larger in the long-run.

4. (14 points) (a) Suppose that 2 L of water at temperature $T_1°C$ is mixed with 3.5 L of water at temperature $T_2°C$. Express the temperature of the resulting mixture in terms of a weighted average.

Total volume of mixture $= 2L + 3.5L = 5.5L$

Temperature of Mixture $= \left(\frac{2}{5.5} T_1 + \frac{3.5}{5.5} T_2\right)°C = \boxed{(0.\overline{36}T_1 + 0.\overline{63}T_2)°C}$

$\boxed{= \left(\frac{4}{11}T_1 + \frac{7}{11}T_2\right)°C}$

(b) Find all equilibria of the discrete-time dynamical system

$$b_{t+1} = \frac{rb_t}{2+b_t},$$

where r is a parameter. Are there any values of r for which there is only one equilibrium?

$$b^* = \frac{rb^*}{2+b^*}$$

$$b^*(2+b^*) = rb^*$$

$$b^*(2+b^*) - rb^* = 0$$

$$2b^* + (b^*)^2 - rb^* = 0$$

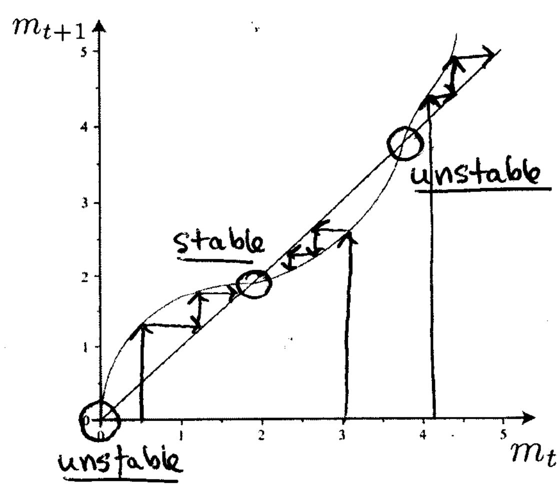
$$(b^*)^2 + (2-r)b^* = 0$$

$$b^*(b^* + 2 - r) = 0 \Rightarrow \boxed{b^* = 0} \text{ or } \boxed{b^* = r-2}$$

When $r = 2$, $r - 2 = 0$, so there is $\boxed{\text{only one equilibrium for } r = 2}$ $\boxed{\& r=0}$

$\boxed{\text{If } r=0, \ b^*=0 \text{ is the only equilibrium.}}$

(c) The updating function for a discrete-time dynamical system $m_{t+1} = f(m_t)$ is graphed below, along with the diagonal. Circle each of the three equilibria, and use *cobwebbing* to help you label each of the equilibria as stable or unstable.

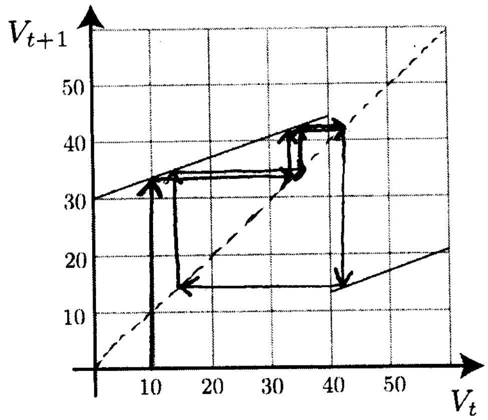


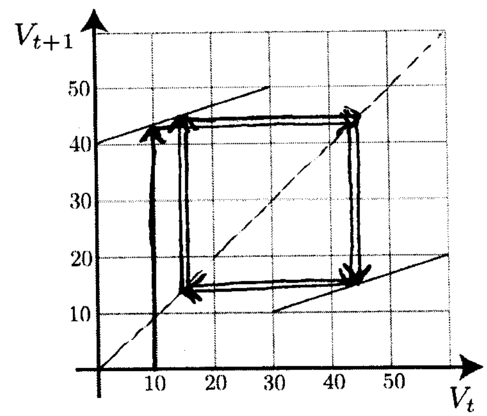
5. (14 points) Let V_t represent the voltage at the AV node in the heart model

$$V_{t+1} = \begin{cases} e^{-\alpha\tau}V_t + u, & \text{if } V_t \le e^{\alpha\tau}V_c \\ e^{-\alpha\tau}V_t, & \text{if } V_t > e^{\alpha\tau}V_c \end{cases}$$

a) For each of the following two graphs of the updating function, cobweb starting from an initial value of $V_0 = 10$, and determine if the heart

i) is healthy, ii) has a 2:1 block, or iii) has the Wenckebach phenomenon.





Wenckebach phenomenon

2:1 block

b) Now let $e^{-\alpha\tau} = 0.25$, $u = 10$, and $V_c = 14$.

i) Does the system have an equilibrium? Justify your answer, and find the equilibrium if there is one.

$$V_{t+1} = \begin{cases} 0.25\,V_t + 10, & V_t \le \frac{1}{0.25}(14) = 4(14) = \boxed{56} \\ 0.25\,V_t, & V_t > \frac{1}{0.25}(14) = \boxed{56} \end{cases}$$

If there is an equilibrium V^*, then

$$V^* = 0.25V^* + 10;$$

$$0.75V^* = 10;$$

$$V^* = 10/0.75 = \frac{40}{3} = \boxed{V^* = 13.\overline{3}}$$

Since $13.\overline{3} < \boxed{56}$, yes, there is an equilibrium.

ii) Recall that $e^{-\alpha\tau} = 0.25$ determines the decay in voltage at the AV node between signals from the SA node. If $\alpha = 0.75$, calculate the time τ between signals. Round to three decimal places.

$$e^{-\alpha\tau} = 0.25$$

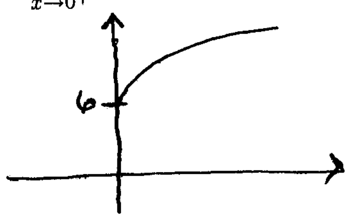
$$e^{-0.75\tau} = 0.25$$

$$\ln\left(e^{-0.75\tau}\right) = \ln(0.25)$$

$$-0.75\tau = \ln(0.25); \quad \tau = \frac{\ln(0.25)}{-0.75} \boxed{\sim 1.848}$$

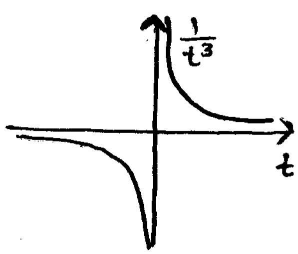
6. (15 points) (a) Find the following limits, if they exist. Show all of your work, and justify your answers to receive full credit. If the limit does not exist, write "DNE," and explain why it does not exist.

i) $\lim_{x \to 0^+} 6\sqrt{x} + 6$ $= \boxed{6}$



ii) $\lim_{y \to 2} \frac{y^2 + 5y + 6}{y + 2}$ $= \lim_{y \to 2} \frac{(y+2)(y+3)}{y+2} = \lim_{y \to 2} (y+3) = 2+3 = \boxed{5}$

iii) $\lim_{t \to 0} \frac{1}{t^3}$



$\lim_{t \to 0^+} \frac{1}{t^3} = \infty$

$\lim_{t \to 0^-} \frac{1}{t^3} = -\infty$

not equal! Therefore, $\boxed{\lim_{t \to 0} \frac{1}{t^3} \text{ DNE}}$

iv) $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$ $= \lim_{\Delta x \to 0} \frac{(x^2 + 2x\Delta x + \Delta x^2) - x^2}{\Delta x}$

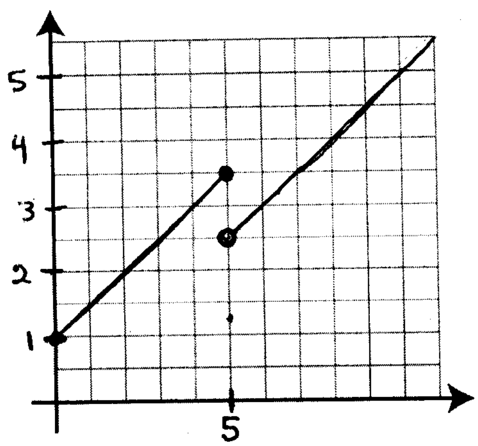
$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x)$

$= \boxed{2x}$

(continuation of problem 6)

(b) Accurately graph the following piecewise function, remembering to label your axes.

$$f(x) = \begin{cases} 0.5x + 1, & \text{if } x \leq 5; \\ 0.5x, & \text{if } x > 5. \end{cases}$$



Find the following limits, if they exist. Show all of your work, and justify your answers.

$\lim_{x\to 5^-} f(x) = 3.5$

$\lim_{x\to 5^+} f(x) = 2.5$

$\lim_{x\to 5} f(x)$ DNE since $\lim_{x\to 5^-} f(x) = 3.5 \neq 2.5 = \lim_{x\to 5^+} f(x)$

Is this function continuous? Why or why not? Use the definition of continuity (*not* a phrase like "the graph can be drawn without lifting the pencil") to justify your answer.

Since $\lim_{x\to 5} f(x)$ DNE,

$f(x)$ is <u>NOT</u> continuous at $x = 5$.

$\boxed{f \text{ is NOT continuous.}}$

7. (15 points) Horace the Hedgehog rolls down a hill, and the distance that he has rolled as a function of time t is given by $f(t) = t^2 + 2t$, where $f(t)$ is in meters, and t is in seconds.

(a) Find a formula for the slope of the secant line that passes through the points $(1, f(1))$ and $(1 + \Delta t, f(1 + \Delta t))$.

$$\text{slope} = \frac{f(1+\Delta t) - f(1)}{\Delta t} = \frac{[(1+\Delta t)^2 + 2(1+\Delta t)] - (1^2 + 2(1))}{\Delta t}$$

$$= \frac{(1 + 2\Delta t + \Delta t^2 + 2 + 2\Delta t) - 3}{\Delta t} = \frac{4\Delta t + \Delta t^2}{\Delta t} = \boxed{4 + \Delta t}$$

(b) Find the average rate of change in f (that is, Horace's average speed) between time $t = 1$ and time $t = 1.5$.

AROC = slope of secant line $\underset{\text{part a}}{\overset{}{=}}$ $4 + \Delta t$

$\Delta t = 1.5 - 1 = 0.5$, so the average rate of change is $4 + 0.5$ $\boxed{= 4.5}$

(c) Find the instantaneous rate of change of f at $t = 1$ *using the limit-definition of the derivative/instantaneous rate of change.*

$$\text{IROC} = \lim_{\Delta t \to 0} (\text{AROC})$$

$$= \lim_{\Delta t \to 0} (4 + \Delta t) = \boxed{4}$$

(d) Find the average rate of change of $f(t)$ between the times t_0 and $t_0 + \Delta t$.

$$\text{AROC} = \frac{f(t_0+\Delta t) - f(t_0)}{\Delta t} = \frac{[(t_0+\Delta t)^2 + 2(t_0+\Delta t)] - [t_0^2 + 2t_0]}{\Delta t}$$

$$= \frac{[t_0^2 + 2t_0\Delta t + \Delta t^2 + 2t_0 + 2\Delta t] - [t_0^2 + 2t_0]}{\Delta t}$$

$$= \frac{2t_0\Delta t + \Delta t^2 + 2\Delta t}{\Delta t} = \frac{\Delta t^2 + (2t_0 + 2)\Delta t}{\Delta t}$$

$$= \Delta t + 2t_0 + 2$$

$$= \boxed{2t_0 + 2 + \Delta t}$$