

NAME: \_\_\_\_\_

SECTION: \_\_\_\_\_ TIME: \_\_\_\_\_

INSTRUCTOR: \_\_\_\_\_

**Instructions:** The exam is closed book and closed notes. You may use an approved calculator, but be sure to show your work on each problem for full credit. Work that is crossed out or erased will not be graded. Turn in any scratch paper that you use during the exam. You will have one hour and 50 minutes to work on the exam.

Problem	Points	Score
1	15	
2	15	
3	14	
4	14	
5	14	
6	14	
7	14	
Total	100	

## CONFIDENTIALITY PLEDGE

I agree that I will not share any information, either specific or general, about the problems on this examination with any other person until the exams have been returned to us in class.

---

(Signature)

1. (15 points) The following table gives data for the number  $c_{t+1}$  of chocolates in a desk at the end of week  $t + 1$  as a function of the number  $c_t$  of chocolates in the desk at the end of week  $t$ .

Number of chocolates at end of week $t$ $c_t$	Number of chocolates at end of week $t + 1$ $c_{t+1}$
100	30
120	40
200	80
250	105

- (a) These data lie on a line—that is,  $c_{t+1}$  is a linear function of  $c_t$ . Find the formula for  $c_{t+1}$  as a function of  $c_t$ .

- (b) Suppose that the number  $C(t)$  of candies in the desk at time  $t$  obeys the equation

$$C(t) = C(0)e^{-0.05t}.$$

Find the half-life of the number of candies.

- (c) If the number of candies obeys the function  $C(t)$  given in part (b), how long does it take for the number of candies to reduce to 20% of the number  $C(0)$  of candies at time 0?

2. (15 points) Two otters splash into their salt-water pool at the Zoo, spilling 67 L of water each time they splash. Their pool usually holds 5900 L of water. If one replaces the 67 L of water that they splash out with 67 L of water that has a concentration of 23 grams/L, the concentration of salt in the pool water may change.

(a) Fill in the blank boxes below to model the situation above. Let  $s_t$  represent the concentration of salt in the pool after the otters have splashed  $t$  times, measured in grams/L. Remember that concentration is equal to the amount of salt (grams) divided by the volume (L).

Step	Volume (L)	Total Salt (grams)	Salt Concentration (grams/L)
H <sub>2</sub> O in pool before otters jump in	5900		$s_t$
Water lost	67		$s_t$
H <sub>2</sub> O in pool after otters jump in			
Water replaced	67		
H <sub>2</sub> O in pool after replacing water			

(b) Write down the discrete-time dynamical system derived from the chart in part (a):

$$s_{t+1} =$$

(c) Suppose that 32 L of water at temperature  $T_1^\circ\text{C}$  is mixed with 17 L of water at temperature  $T_2^\circ\text{C}$ . Express the temperature of the resulting mixture in terms of a weighted average.

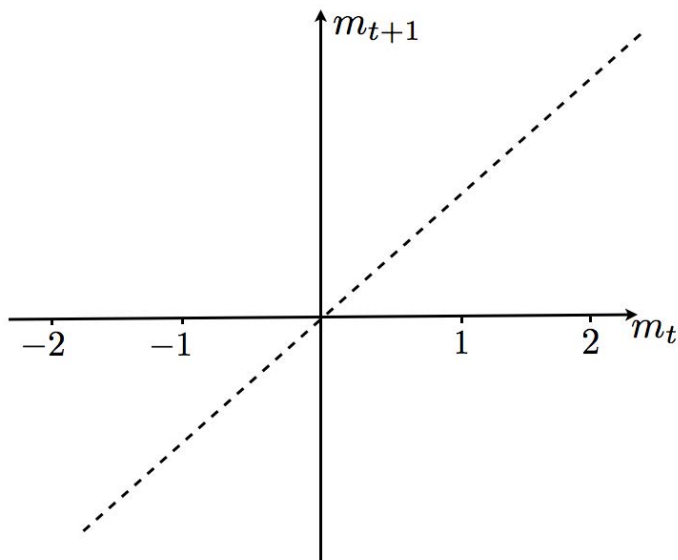
3. (14 points) (a) Write down a discrete-time dynamical system and an initial condition to describe the following situation: A lake is created at the zoo and stocked with 780 crayfish. Every year from then on, the otters catch 23% of the crayfish in the lake. The crayfish do not reproduce, but at the end of the year the lake is restocked by adding 540 crayfish. (Let  $C_t$  = the number of crayfish in the lake at the end of year  $t$ .)

- (b) Find all equilibria of the discrete-time dynamical system

$$p_{t+1} = \frac{3p_t}{2p_t - k},$$

where  $k$  is a parameter. For what values of  $k$  is there a positive equilibrium?

- (c) Consider the discrete-time dynamical system  $m_{t+1} = m_t^5$ . i) *Graph* the updating function on the axes below (the diagonal  $m_{t+1} = m_t$  is already graphed). ii) *Circle* ALL of the equilibria, and use *cobwebbing* to help you label each of the equilibria as stable or unstable.



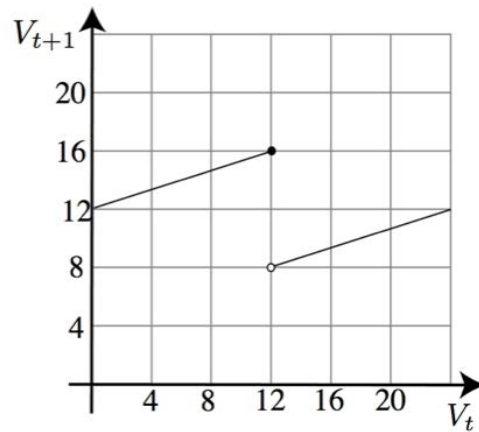
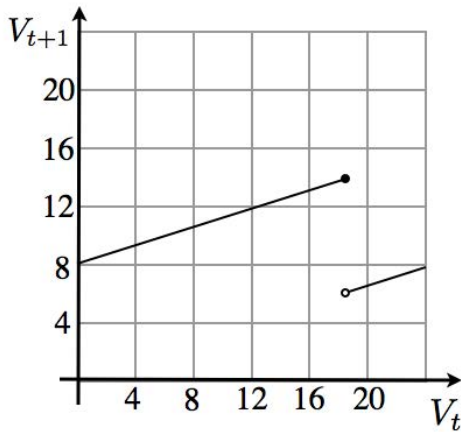
4. (14 points) Let  $V_t$  represent the voltage at the AV node in the heart model

$$V_{t+1} = \begin{cases} e^{-\alpha\tau}V_t + u, & \text{if } V_t \leq e^{\alpha\tau}V_c \\ e^{-\alpha\tau}V_t, & \text{if } V_t > e^{\alpha\tau}V_c \end{cases}$$

a) For each of the following two graphs of the updating function, cobweb for at least 4 steps starting from an initial value of  $V_0 = 20$ , and determine if the heart

i) is healthy, ii) has a 2:1 block, or iii) has the Wenckebach phenomenon.

Include arrows on your cobweb diagram.



b) Now let  $e^{-\alpha\tau} = 0.6$ ,  $u = 5$ , and  $V_c = 20$ . Does the system have an equilibrium? Justify your answer algebraically (that is, without drawing a graph), and find the equilibrium if there is one.

5. (14 points) (a) Find the following limits, if they exist. Show all of your work, and justify your answers to receive full credit. If a limit does not exist, write “DNE,” and explain why it does not exist.

i)  $\lim_{x \rightarrow 0} \frac{\tan x}{x + 2}$

ii)  $\lim_{s \rightarrow -6} \frac{s^2 + 3s - 18}{s + 6}$

iii)  $\lim_{x \rightarrow 2} \frac{1}{(x - 2)^7}$

iv)  $\lim_{t \rightarrow 3} \frac{1}{(t - 3)^2}$

6. (14 points) (a) We are interested in the density of a substance at a temperature of absolute zero (which is 0 Kelvin). However, we cannot measure the density directly at 0 Kelvin because it is impossible to reach absolute 0. Instead, we measure density for small values of the temperature.

Suppose that density  $D$  is a function of temperature  $T$  (measured in Kelvin) according to  $D(T) = \frac{100}{1+2T}$ .

i) What is  $\lim_{T \rightarrow 0^+} \frac{100}{1+2T}$ ?

- ii) How close to 0 Kelvin would the temperature have to be for the density to be within 1% of the limit?

(b) Consider the function

$$f(x) = \begin{cases} \sin(x), & \text{if } x < \pi; \\ 0, & \text{if } x = \pi; \\ \cos(x) + 1 & \text{if } x > \pi. \end{cases}$$

Find the following limits, if they exist. If a limit does not exist, write “DNE,” and explain why it does not exist.

$$\lim_{x \rightarrow \pi^-} f(x)$$

$$\lim_{x \rightarrow \pi^+} f(x)$$

$$\lim_{x \rightarrow \pi} f(x)$$

Is this function continuous at  $x = \pi$ ? Why or why not? Use the definition of continuity (*not* a phrase like “the graph can be drawn without lifting the pencil”) to justify your answer.

7. (14 points) Hildebrünn throws a ball up into the air from the top of a tower. Suppose that the height  $h(t)$  (in meters) of the ball as a function of time  $t$  (in seconds) is given by  $h(t) = -5t^2 + 30t + 20$ .

(a) Find a formula for the slope of the secant line that passes through the points  $(2, h(2))$  and  $(2 + \Delta t, h(2 + \Delta t))$ . Simplify your answer.

(b) Find the average rate of change in  $h$  between time  $t = 2$  and time  $t = 2.5$ .

(c) Find the instantaneous rate of change of  $h$  at  $t = 2$  *using the limit-definition of the derivative/instantaneous rate of change*. Is the height of the ball increasing or decreasing at time  $t = 2$ ?

(d) Find the equation for the tangent line to the graph of  $h(t)$  at the point  $(2, h(2))$ .