Fall 2017 answer key.

1. (16 points) Compute the derivatives of the following functions. You do NOT need to simplify your answer. Use parentheses to indicate multiplication where appropriate, and make sure that your notation is correct.

(a) $f(x) = x^3 + e^{3x}$ power rule + exponential + chain rule

$f'(x) = 3x^2 + e^{3x}(3)$

(b) $f(t) = \overset{u}{t^3}\,\overset{v}{\ln(t)}$ product rule

$f'(t) = uv' + vu'$

$f'(t) = t^3\left(\frac{1}{t}\right) + \ln(t)(3t^2)$

(c) $k(x) = (\underset{u}{\sin(x)}\underset{v}{e^x})^3$ power rule, chain rule, product rule

$k'(x) = 3(\sin(x)e^x)^2 \cdot [(\sin(x))(e^x) + (e^x)(\cos(x))]$

(d) $r(t) = \cos(c(t^5 - 3c))$, where c is a constant.

simplify first by distributing c

$r(t) = \cos(ct^5 - \underset{\text{constant}}{3c^2})$ chain rule, power rule

$r'(t) = -\sin(ct^5 - 3c^2)(5ct^4)$

2. (11 points) Consider the discrete time dynamical system $x_{t+1} = 2.5x_t(1 - x_t)$.

(a) (4 points) Find the equilibrium for the DTDS.

$x^* = 2.5x^*(1 - x^*)$

Divide by x^* and remember to check if $x^* = 0$ is a solution.

$1 = 2.5(1 - x^*)$

$\frac{1}{2.5} = (1 - x^*)$

$x^* = 1 - \frac{1}{2.5}$

$x^* = 1 - \frac{1}{5/2}$

$x^* = 1 - 2/5$

$x^* = 3/5$ and $x^* = 0$ are equilibrium.

Check $\left(\frac{5}{2}\right)(3/5)(1 - 3/5) = \left(\frac{3}{2}\right)\left(\frac{2}{5}\right) = 3/5$ ✓

(b) (4 points) Is the non-zero equilibrium stable or unstable? Justify your answer using the Stability Theorem.

$f(x_t) = 2.5x_t - 2.5x_t^2$

$f'(x_t) = 2.5 - 5x_t$

$f'(3/5) = 2.5 - 5(3/5)$

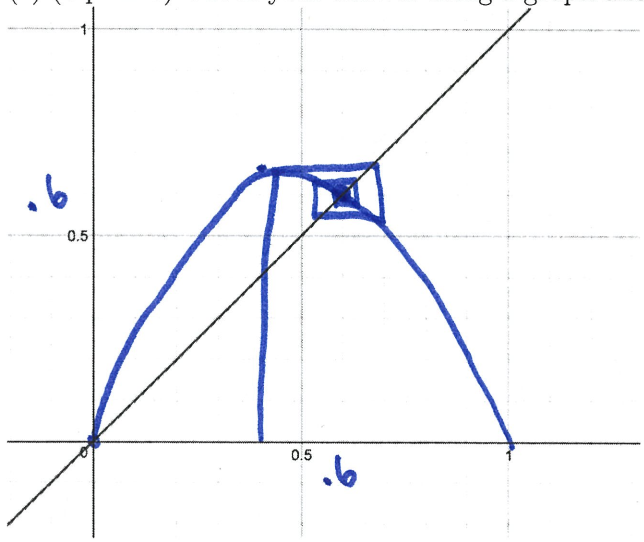
$= 2.5 - 3$

$= -.5$

$|f'(3/5)| = |.5| < 1$

The rate of change of updating function is between -1 and 1 so $x^* = 3/5$ is stable.

(c) (3 points) Check your answer using a graph and cobwebbing. Label your axes.



Arbitrarily choose initial value of $X_0 = .4$

The equilibrium is at .6 which is 3/5.

Cobwebbing shows equilibrium is stable

3. (10 points) Consider the logistic growth model.

$$x_{t+1} = 3.2rx_t(1 - x_t)$$

(a) What condition on r guarantees that the equilibrium $x^* = 0$ is stable? Justify your answer with the Stability Theorem. Use correct notation in your work.

$f(x_t) = 3.2rx_t - 3.2rx_t^2$

$f'(x_t) = 3.2r - 6.4rx_t$

$f'(0) = 3.2r - 0$

When $|3.2r| < 1$ the equilibrium $x^* = 0$ is stable.

$-1 < 3.2r < 1$

$\boxed{-1/3.2 < r < 1/3.2}$

(b) Find the non-zero equilibrium in terms of parameter r.

$x^* = 3.2rx^*(1 - x^*)$

divide by x^*, remember that you lose solution $x^* = 0$.

$1 = 3.2r(1 - x^*)$

$\frac{1}{3.2r} = 1 - x^*$

$\frac{1}{3.2r} - 1 = -x^*$

$1 - \frac{1}{3.2r} = x^*$

4. (9 points)

A ball is thrown vertically upward from the roof of a 12 foot building with a velocity of 85 feet per second. The ball's height above the ground after t seconds is given by the function

height function

$$s(t) = 12 + 85t - 16t^2$$

(a) What is the maximum height the ball reaches? Find and justify your answer using properties of the derivative of the function. Check your answer with a graphing calculator.

$$S'(t) = 85 - 32t$$

$$85 - 32t = 0$$

$$85 = 32t$$

$$t = \frac{85}{32} = 2.65625 \text{ sec}$$

time ball reaches maximum height

plug into height function

$$S\left(\frac{85}{32}\right) = 12 + 85\left(\frac{85}{32}\right) - 16\left(\frac{85}{32}\right)^2 = \boxed{124.89 \text{ ft}}$$

(b) Suppose:

$$s'(t_1) > 0$$

According to the model is the ball going up or down after t_1 seconds? Justify your choice using the meaning of a derivative.

S' represents the rate of change of the function S. Because the rate of change is positive, the ball is going up. This is because the problem defined the building as positive 12 ft tall and the velocity is positive (moving up).

(c) What is the velocity of the ball when it hits the ground? The ground has height zero.

· Find zeros of $s(t)$

$S(t) = 12 + 85t - 16t^2 = 0$ use calculator or quadratic formula

$t = 0$ and $t = 5.45$ sec

↳ this is the value when the ball hits the ground

· velocity $= s'(t)$

$S'(t) = 85 - 32t \longrightarrow S'(5.45) = 85 - 32(5.45)$

$$\boxed{\text{velocity} = -89.4 \text{ ft/sec}}$$

5. (12 points) Suppose the function $M(t)$ gives the mass of an insect(in grams) and the function V(t) gives the volume of an insect in cubic centimeters. Assume $t > 0$. Let $M(t) = 3t$ and $V(t) = 2 + t^2$.

(a) Density is computed by comparing the mass of an object to its volume.

$$Density = \frac{Mass}{Volume}$$

If the mass of an insect increases while the volume of the insect stays constant what happens to the density of the insect?

The density would increase because of the proportional relationship between mass and density.

(b) Write a formula giving the density ($D(t)$) of the insect at time t with mass $M(t)$ and volume $V(t)$.

$$D(t) = \frac{M(t)}{V(t)} = \frac{3t}{2+t^2}$$

(c) Find the positive value of t where the rate of change of the insect's density is zero. This value of t is also called a critical point.

$$D'(t) = \frac{(2+t^2)(3) - 3t(2t)}{(2+t^2)^2}$$

Since a fraction will equal Ø when the numerator is zero, we can set the numerator equal to Ø to find our critical point.

$$0 = (2+t^2)(3) - 3t(2t)$$

$0 = -6t^2 + 3t^2 + 6$

$0 = -3t^2 + 6$

$-6 = -3t^2$

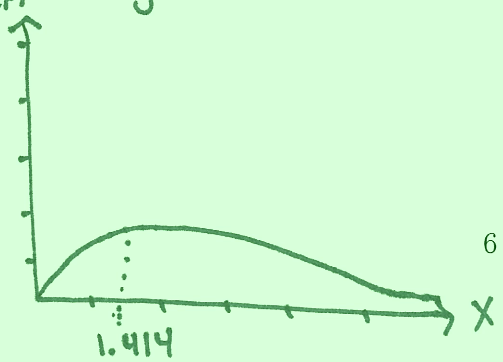
$2 = t^2$

$\sqrt{2} = t$

we want the positive value, so we get $\boxed{t = 1.414}$

(d) Consider the value of time t you found in part (c). Does this point correspond to a local (relative) maximum or local (relative) minimum value of density or neither? Justify using graph or derivative properties.

If we graph the original equation, f(x) we get:



The point at ~~[illegible]~~ x = 1.414 is higher than everything else around it, which makes it a local (relative) maximum.

6. (10 points) Consider the discrete-time dynamical system

$$T_{t+1} = 2(1 - T_t)T_t - hT_t$$

describing a population of fish being harvested at rate h, where $0 \le h \le 1$.

(a) Find the nonzero equilibrium population T^* as a function of h.

$T^* = 2(1-T^*)T^* - hT^*$

$T^* = T^*[2(1-T^*) - h]$ divide both sides by T^*

don't forget that you lose the $T^* = 0$ solution in this step

$1 = 2(1-T^*) - h \rightarrow 1 = 2 - 2T^* - h$

$2T^* = 1 - h$

$\boxed{T^* = \frac{1-h}{2}}$

(b) The equilibrium harvest is given by $P(h) = hT^*$, where T^* is the equilibrium you found in part (a). Find the value of h that maximizes $P(h)$ on the interval $0 \le h \le 1$. Use ***derivative properties*** to justify your answer.

$P(h) = hT^* \rightarrow P(h) = h\left(\frac{1-h}{2}\right) \rightarrow P(h) = \frac{h-h^2}{2} = \frac{1}{2}(h-h$

$P'(h) = \frac{1}{2}(1-2h) \rightarrow \frac{1}{2} - h = P'(h)$

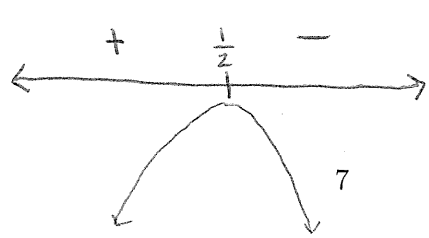
- maximum occurs when $P'(h) = 0$

$0 = \frac{1}{2} - h \rightarrow \boxed{h = \frac{1}{2}}$

- to prove that $h = \frac{1}{2}$ is a maximum (rather than a minimum or inflection point), check points to the left + right of $\frac{1}{2}$

$P'(\frac{1}{4}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} > 0$

$P'(\frac{3}{4}) = \frac{1}{2} - \frac{3}{4} = \frac{-1}{4} < 0$



Because the rate of change goes from positive to negative at $h = \frac{1}{2}$, it is a maximum.

7. (12 points) Evaluate the following limits. Show all of your work. If you use leading behavior, justify your answer by explaining all of your steps. If you use L'Hopital's Rule, justify why it can be applied each time you use it.

(a) $\lim_{x\to\infty} \frac{x^3+3}{\ln(x)+2x}$ $= \frac{\infty}{\infty}$ → use L'Hopital's rule

$\lim_{x\to\infty} \frac{3x^2}{\frac{1}{x}+2} = \frac{\infty}{0+2} = \frac{\infty}{2} = \boxed{\infty}$

(b) $\lim_{x\to 0} \frac{\sin(x)}{e^{3x}+10x^{-2}}$ $= \frac{0}{1+\frac{10}{0}} = \frac{0}{1+\infty} = \boxed{0}$

* you cannot use L'Hopital's rule because there is not an indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$

(c) $\lim_{x\to\infty} \frac{e^x+5x^3+x^{-1}}{\ln(x)+2x^3}$ $= \frac{\infty}{\infty}$ → use L'Hopital's rule

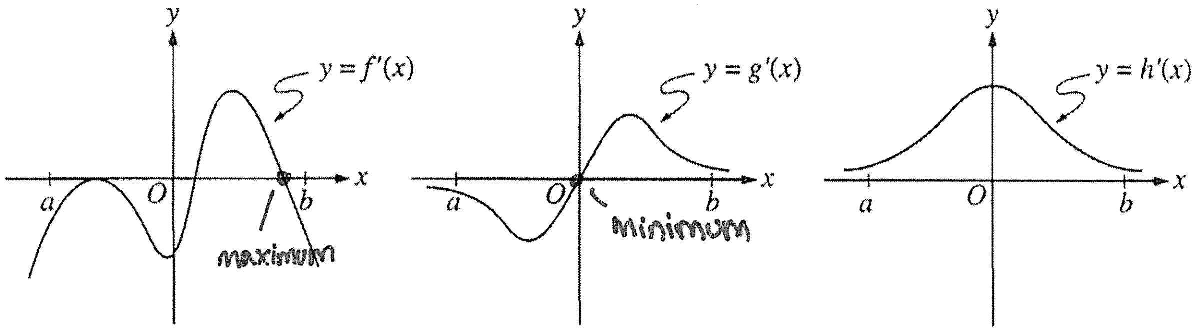
$x^{-1} \nearrow \frac{1}{x}$

$\lim_{x\to\infty} \frac{e^x+15x^2-x^{-2}}{\frac{1}{x}+6x^2} = \frac{\infty}{\infty}$ → use L'Hopital's rule

$\lim_{x\to\infty} \frac{e^x+30x+2x^{-3}}{-x^{-2}+12x} = \frac{\infty}{\infty}$ → use L'Hopital's rule

$\lim_{x\to\infty} \frac{e^x+30-6x^{-4}}{2x^{-3}+12} = \frac{\infty+30-0}{0+12} = \boxed{\infty}$

8.

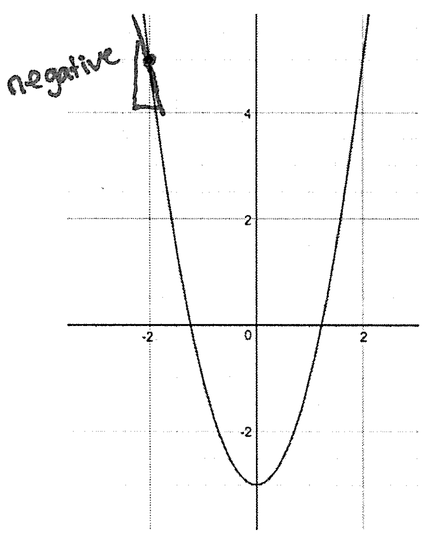


(5 points) The graphs of the **derivatives** of the function f, g, and h are shown above. Which of the functions f, g, or h have a relative maximum(local maximum) on the open interval $a < x < b$?

A) f only
B) g only
C) h only
D) f and g only
E) f, g, and h

Because the graphs are graphs of derivatives, maximums + minimums correspond to where the function crosses the x axis. The only function that changes from increasing to decreasing as it crosses the x-axis is f'(x), which means f(x) is the only function with a maximum on the interval.

9. (5 points) The graph of $y = f(x)$ is given below.

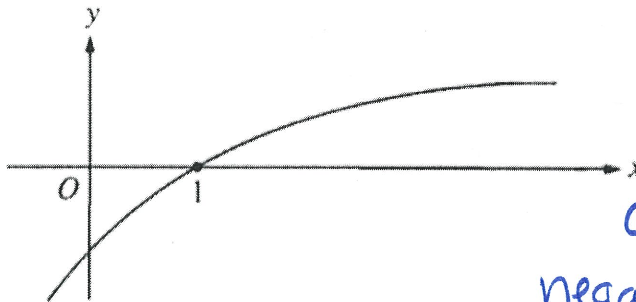


The value of $f'(-2)$ is:

A) Positive
B) Negative
C) Undefined
D) Zero
E) Not enough information.

The key here is to recognize that the graph is f(x) and the question is asking about f'(x). Even though the value of f(x) is positive, the rate of change at that point is negative which makes f'(x) negative. We can demonstrate this using tangent lines.

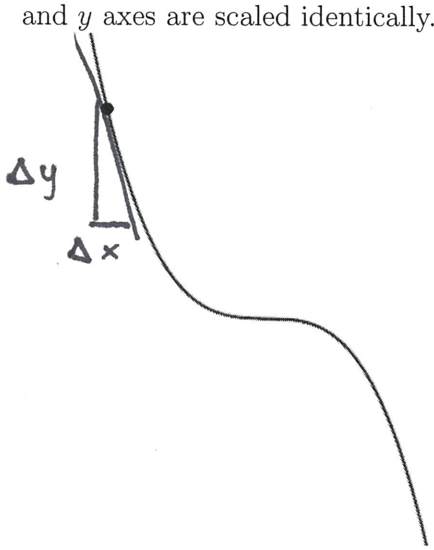
At $x=1$ The function is 0. So $f(1)=0$.
at $x=1$, $f(x)$ is increasing so $f'(1)$ is positive.
At $x=1$ $f(x)$ is concave down so $f''(1)$ is negative.
negative < 0 < positive.



10. (5 points) The graph of a twice-differentiable function f is shown in the figure above. Which of the following is true?

A) $f(1) < f'(1) < f''(1)$
B) $f(1) < f''(1) < f'(1)$
C) $f'(1) < f(1) < f''(1)$
D) $f''(1) < f(1) < f'(1)$
E) $f''(1) < f'(1) < f(1)$

11. (5 points) Estimate the slope of the function at the point shown. See below. The x and y axes are scaled identically.



Even though there is no axis, we can make reasonable estimates of the slope by using tangent lines. First, we know the slope is negative because our tangent line is going down, which eliminates all of the positive options. This is also a steep line so we can eliminate the fractional answer because the slope isn't that small. Finally, we can eliminate the -20 answer because on the triangle we make from our tangent line, our "Δx" side won't fit 20 times into our "Δy" side. If we had to guess, it fits ≈3-4 times, which when combined with our other information makes -4 the best answer.

A) The slope is approximately 2
B) The slope is approximately -4
C) The slope is approximately -20
D) The slope is approximately 4
E) The slope is approximately $\frac{-1}{4}$