1. (14 points) Compute the derivatives of the following functions. <u>You do NOT need to simplify your answer.</u> Use parentheses to indicate multiplication where appropriate, and make sure that your notation is correct.

(a) $f(x) = \ln(x) + 4x^3$

$$f(x) = \underline{\ln(x)} + 4x^{③}$$

$$f'(x) = \frac{1}{x} + 12x^2$$

Special Case

$$f(x) = \ln(x)$$

$$f'(x) = \frac{1}{x}$$

1. (14 points) Compute the derivatives of the following functions. You do NOT need to simplify your answer. Use parentheses to indicate multiplication where appropriate, and make sure that your notation is correct.

(b) $p(x) = \underbrace{(mx+b)}_{f}\,\underbrace{\cos(x)}_{g}$, $\boxed{\text{where } m \text{ and } b \text{ are constants.}}$

$$p'(x) = \overbrace{(mx+b)}^{f}\,\overbrace{(-\sin(x))}^{g'} + \overbrace{\cos(x)}^{g}\,\overbrace{(m)}^{f'}$$

Product Rule

$$\underbrace{(\text{something})}_{f}\underbrace{(\text{something})}_{g}$$

$$fg' + gf'$$

$g = \cos(x)$

$g' = -\sin(x)$

$f = mx + b$

$\boxed{f' = m}$

$$\boxed{\begin{gathered} f = \overset{m}{5}x + \overset{b}{7} \\ f' = 5 \end{gathered}}$$

1. (14 points) Compute the derivatives of the following functions. You do NOT need to simplify your answer. Use parentheses to indicate multiplication where appropriate, and make sure that your notation is correct.

(c) $f(x) = \boxed{e^{\boxed{\sin(x)}}}$

$f'(x) = ((\cos(x))e^{(\sin(x))}$

Chain Rule

$(\underline{DIN})(DOUT)$

$\rightarrow IN = \sin(x)$

$DIN = \cos(x)$

$OUT = e^{(\sin(x))}$

$\rightarrow DOUT = e^{(\sin(x))}$

Special Case

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f(x) = e^{\boxed{x}}$

$f'(x) = e^{x}$

More than just an $\underline{X}$, it's Chain Rule

1. (14 points) Compute the derivatives of the following functions. You do NOT need to simplify your answer. Use parentheses to indicate multiplication where appropriate, and make sure that your notation is correct.

(d) $r(x) = \frac{e^x}{x^2+\frac{x}{7}}$

$$\frac{1x}{7} = \frac{1}{7}x$$

$$r(x) = \frac{e^x}{x^2 + \frac{1}{7}x} \quad \frac{\text{hi}}{\text{low}}$$

$$r'(x) = \frac{(x^2+\frac{1}{7}x)(e^x)-(e^x)(2x+\frac{1}{7})}{(x^2+\frac{1}{7}x)^2}$$

Quotient Rule

$$\frac{\text{Something hi}}{\text{Something low}}$$

$$\frac{(\text{low})(\underline{\text{dhi}})-(\text{hi})(\underline{\text{dlow}})}{(\text{low})^2}$$

Special Case

$f(x) = e^x$

$f'(x) = e^x$

2. Consider the discrete time dynamical system $\boxed{x_{t+1} = 1.2x_t(1 - x_t).}$

(a) (3 points) Algebraically find all of the equilibrium for the DTDS.

$$x^* = 1.2x^*(1 - x^*)$$

$$x^* = 1.2x^* - 1.2x^{*2}$$

$$-x^* \qquad -x^*$$

$$0 = .2x^* - 1.2x^{*2}$$

Factor

$$0 = \underline{.2x^*}(1 - 6x^*)$$

$$\frac{.2x^*}{.2} = \frac{0}{.2}$$

$$\boxed{x^* = 0}$$

$$1 - 6x^* = 0$$

$$+6x^* \qquad +6x^*$$

$$\frac{1}{6} = \frac{6x^*}{6}$$

$$\boxed{\frac{1}{6} = x^*}$$

2. Consider the discrete time dynamical system $x_{t+1} = 1.2x_t(1 - x_t)$.

(b) (3 points) Is the non-zero equilibrium stable or unstable? Justify your answer using the Stability Theorem. To get full credit you have to explicitly use the Stability Theorem.

~~$x^* = 0$~~ $\boxed{x^* = \frac{1}{6}}$

$$\longrightarrow f(x) = 1.2x(1-x)$$

$$f(x) = 1.2x - 1.2x^2$$

$$\longrightarrow f'(x) = 1.2 - 2.4x$$

$$\longrightarrow f'\left(\tfrac{1}{6}\right) = 1.2 - 2.4\left(\tfrac{1}{6}\right)$$

$$\longrightarrow f'\left(\tfrac{1}{6}\right) = .8$$

$$\longrightarrow |.8| < 1$$

$\boxed{\text{Stable} \quad x^* = \frac{1}{6}}$

$|f'(eq)| < 1$ Stable

$|f'(eq)| > 1$ unstable

$|f'(eq)| = 1$ Test fails

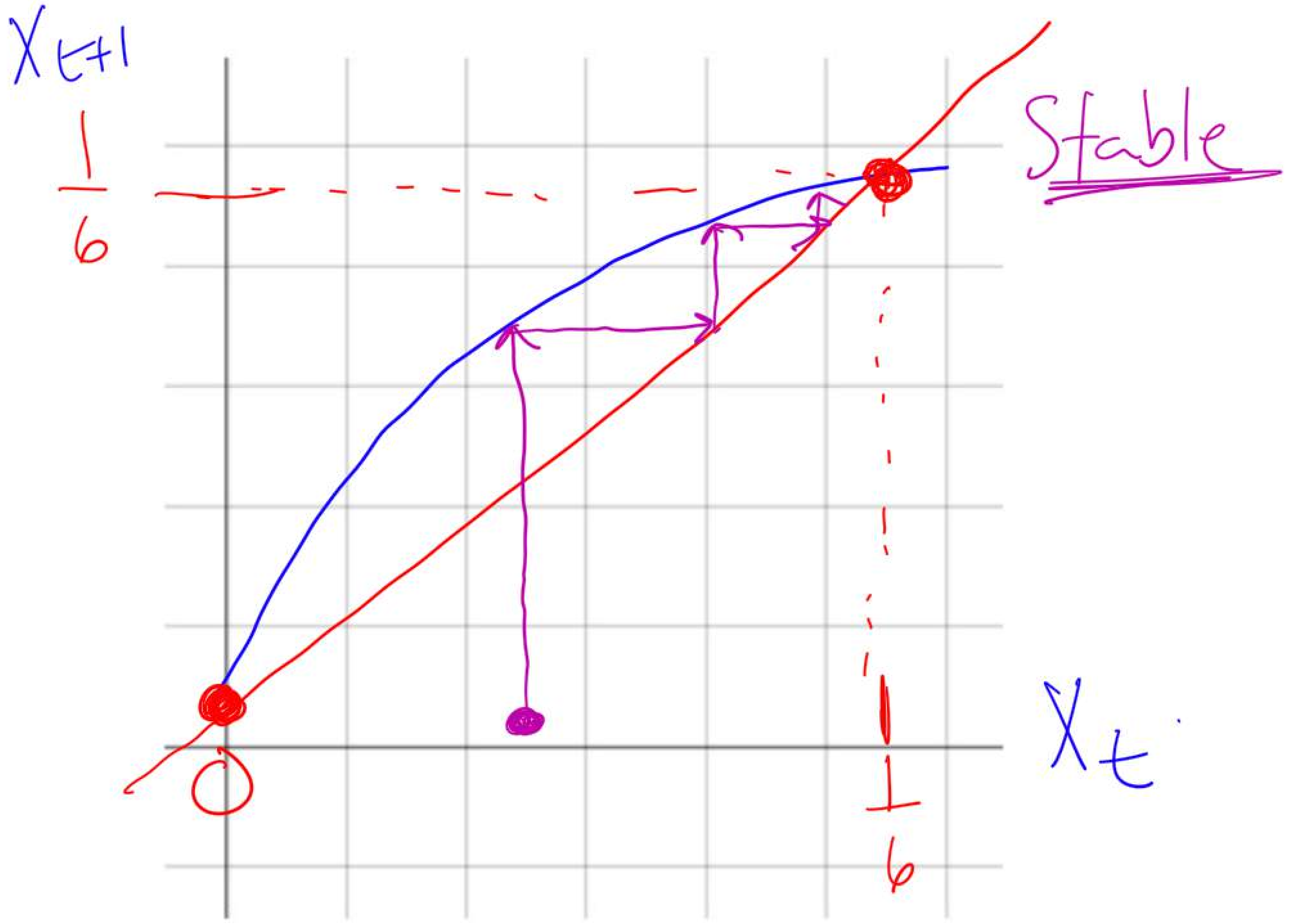
2. Consider the discrete time dynamical system $\underset{Y}{\boxed{x_{t+1}}} = \underset{\uparrow X}{\boxed{1.2x_t(1-x_t).}} \longrightarrow Y_1 = 1.2x(1-x)$

(c) (3 points) Use a graphical method to check your answer for part a. Clearly label axis. Make it clear how the graph is being used to check part a.

$Y_2 = X$ Equilibrium Line

$\underline{X^\# = 0}$

$\underline{X^\# = \frac{1}{6}}$



Look for intersects

3. (9 points) Consider the logistic model.

$$x_{t+1} = \boxed{r x_t (1 - x_t)}$$

(a) Assume parameter r is greater than 0.
What condition on r guarantees that the equilibrium $x^* = 0$ is stable?
Justify your answer with the Stability Theorem. Use correct notation in your work.

$$f(x) = rx(1-x)$$

$$f(x) = rx - rx^2$$

$$f'(x) = r - 2rx$$

$$f'(0) = r - \underline{2r(0)}$$

$$* f'(0) = r$$

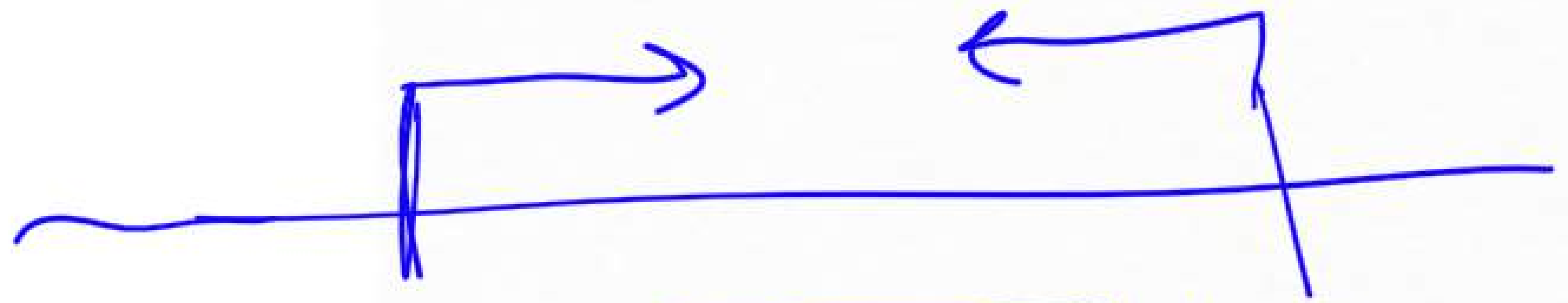
$$|r| < 1$$

$$-1 < r < 1$$

$$\boxed{0 < r < 1}$$

Stable

$$|f'(eq)| < 1$$



$$-1 < \square < 1$$

3. (9 points) Consider the logistic model.

$$\longrightarrow x_{t+1} = rx_t(1 - x_t) \quad \boxed{> 0} \quad < 0$$

(b) Find the positive, non-zero equilibrium in terms of parameter r. After you find the equilibrium in terms of r indicate which values of r make it positive.

$$x^* = rx^*(1-x^*)$$

$$x^* = rx^* - rx^{*2}$$

$$-x^* \qquad -x^*$$

$$0 = rx^* - x^* - rx^{*2}$$

$$0 = x^*(r - 1 - rx^*)$$

$$\cancel{x^* = 0}$$

$$\boxed{r - 1 - rx^* = 0}$$

$$r - 1 - rx^* = 0$$

$$-r \qquad -r$$

$$-1 - rx^* = -r$$

$$+1 \qquad +1$$

$$\frac{\cancel{-r}x^*}{\cancel{-r}} = \boxed{\frac{-r}{-r}} \boxed{+\frac{1}{-r}}$$

$$\boxed{x^* = 1 - \frac{1}{r}}$$

$$1 - \frac{1}{r} > 0$$

$$+\frac{1}{r} \qquad +\frac{1}{r}$$

$$r \cdot 1 > \frac{1}{\cancel{r}} \cdot r$$

$$\longrightarrow \boxed{r > 1}$$

3. (9 points) Consider the logistic model.

$$x_{t+1} = rx_t(1 - x_t)$$

$$x^\# = \boxed{1 - \frac{1}{r}}$$

$$-2\cancel{r} \cdot -\frac{1}{\cancel{r}} = 2$$

(c) For what values of r is the non-zero equilibrium stable? Justify your answer with the Stability Theorem.

$$|f'(e_\xi)| < 1 \text{ Stable}$$

$$f(x) = rx(1-x)$$

$$f(x) = rx - rx^2$$

$$f'(x) = r - 2rx$$

$$f'\left(1 - \frac{1}{r}\right) = r \boxed{-2r}\left(1 - \frac{1}{r}\right)$$

$$= r - 2r + 2$$

$$f'\left(1 - \frac{1}{r}\right) = -r + 2$$

$$|-r+2| < 1$$

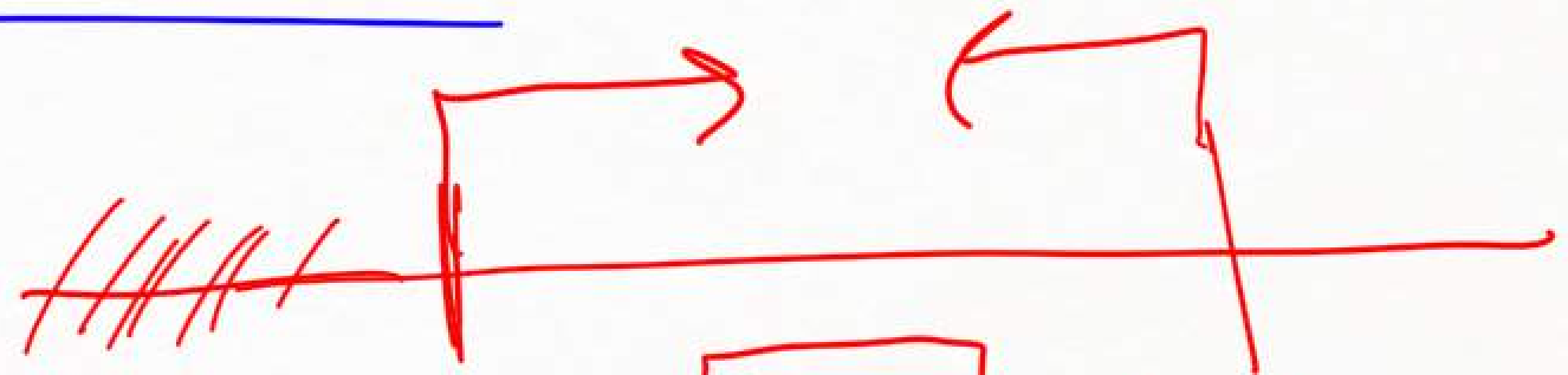
$$-1 < -r + 2 < 1$$

$$-2 \qquad -2 \qquad -2$$

$$-3 < -r < -1$$

$$\div -1 \qquad \div -1 \qquad \div -1$$

$$\boxed{3 > r > 1}$$



$$-1 < \square < 1$$

4. (5 points) Solve the following inequality:

$$|5-x| > 3$$

$5-x < -3$ OR

$-5 \qquad -5$

$-x < -8$

$\overline{-1} \qquad \overline{-1}$

$\boxed{x > 8}$

OR

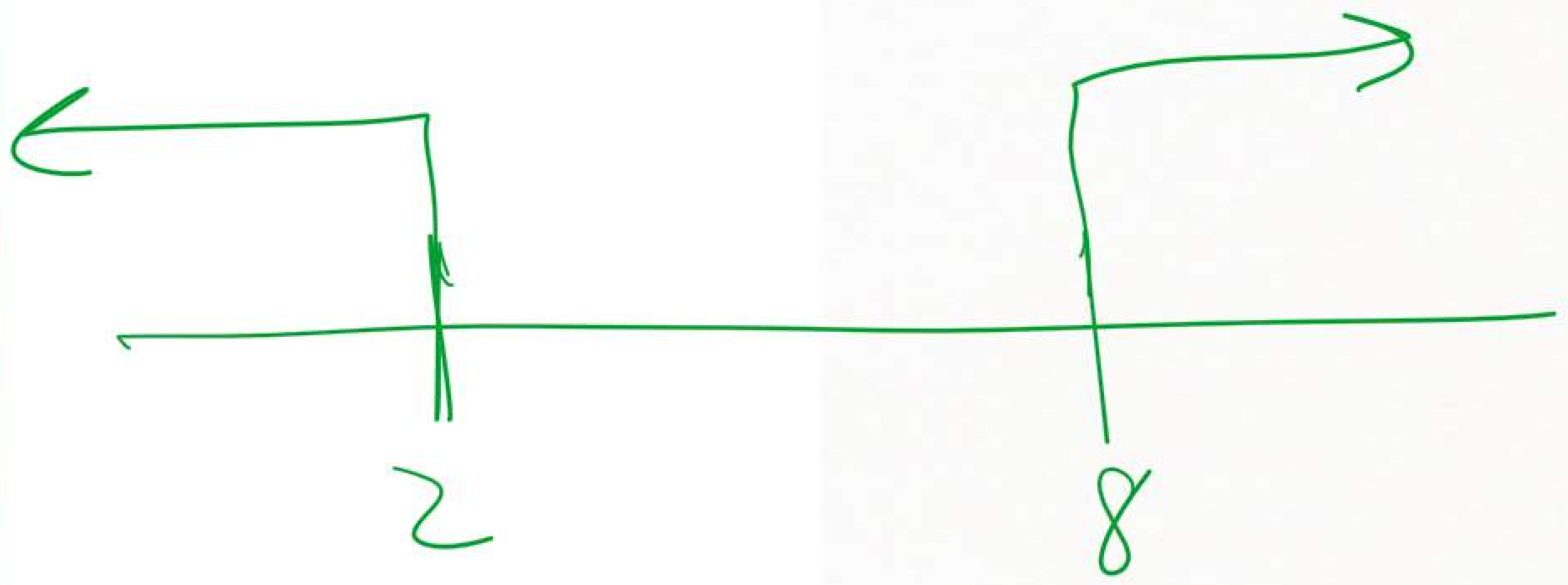
$5-x > 3$

$-5 \qquad -5$

$-x > -2$

$\overline{-1} \qquad \overline{-1}$

$\boxed{x < 2}$



Suggested way to check: Graph your solution on a number line and check if the values you shaded make the inequality true.(Not required.)

5. (8 points) A population of aliens with unlimited resources grows according to the following model (t stands for time):

$$\frac{5t}{5}$$

$$\longrightarrow P(t) = 100e^{\boxed{\ln(1.02)t}}$$

IN = ln(1.02)t

$\longrightarrow$ DIN = ln(1.02)

OUT = 100e( )

DOUT = 100e( )

Special Case

$f(x) = e^{(x)}$

$f'(x) = e^{(x)}$

Anything More than Just X

Chain Rule

(DIN)(DOUT)

a) What is the rate of change of the alien's population with respect to time?

$$P'(t) = \ln(1.02)(100e^{\ln(1.02)t})$$

As t gets bigger

up

b) As time passes, is the rate of change of the alien's population increasing or decreasing? Justify using any method.

Down

$$P'(t) = \ln(1.02)100e^{\ln(1.02t)}$$

$$\longrightarrow P'(1) = 2.019$$

$$\longrightarrow P'(10) = 2.41$$

Increasing

6. (8 points) Consider the discrete-time dynamical system

$$\longrightarrow T_{t+1} = 2(1 - T_t)T_t - hT_t$$

describing a population of fish being harvested at rate h, where $0 \leq h \leq 1$.

(a) Find the $\boxed{\text{nonzero equilibrium}}$ population T^* as a function of h.

$$T^* = 2(1-T^*)T^* - hT^*$$

$$T^* = (2-2T^*)T^* - hT^*$$

$$T^* = 2T^* - 2T^{*\textcircled{2}} - hT^*$$

$$-T^* \qquad -T^*$$

$$0 = \underline{T^*} - \underline{2T^{*2}} - \underline{hT^*}$$

$$0 = T^*(1-2T^*-h)$$

$$\cancel{T^*=0} \qquad \hookrightarrow 1-2T^*-h=0$$

$$\longrightarrow 1-2T^*-h=0$$

$$+h \quad +h$$

$$1-2T^*=h$$

$$-1 \qquad -1$$

$$* \longrightarrow \frac{-2T^*}{-2} = \frac{h}{-2} \boxed{-\frac{1}{-2}}$$

$$\longrightarrow \boxed{T^* = \boxed{\frac{h}{-2}} + \frac{1}{2}}$$

$$\longrightarrow \boxed{T^* = -\frac{1}{2}h + \frac{1}{2}}$$

6. (8 points) Consider the discrete-time dynamical system

$$T_{t+1} = 2(1 - T_t)T_t - hT_t$$

describing a population of fish being harvested at rate h, where $0 \leq h \leq 1$.

(a) Find the nonzero equilibrium population T^* as a function of h. $T^* = -\frac{1}{2}h + \frac{1}{2}$

(b) The equilibrium harvest is given by $\boxed{P(h) = hT^*}$, where T^* is the equilibrium you found in part (a). Find the value of h that maximizes $P(h)$ on the interval $0 \leq h \leq 1$. Use any method to justify answer(including calculator).

1st Derivative Test

S1] Find $f'(x)$

S2] Find Critical Values

$f'(x) = 0$ OR $f'(x) = \text{DNE}$

fractions

S3] Do the 1st Derivative (#Line Game)

$$P(h) = h\left(-\frac{1}{2}h + \frac{1}{2}\right)$$

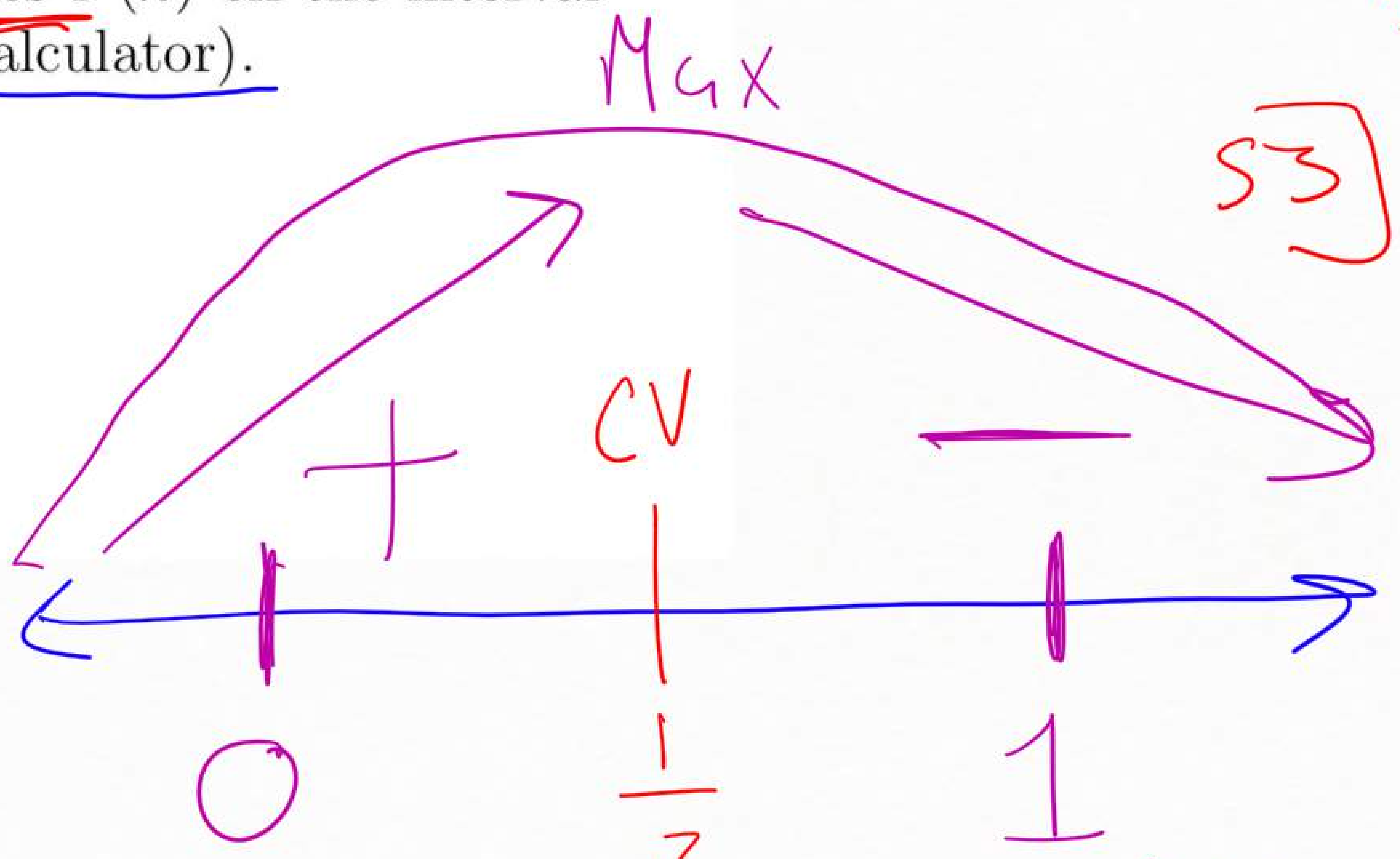
$$P(h) = -\frac{1}{2}h^2 + \frac{1}{2}h$$

$$P'(h) = -1h + \frac{1}{2}$$

$$0 = -1h + \frac{1}{2}$$

$$+h \qquad +1h$$

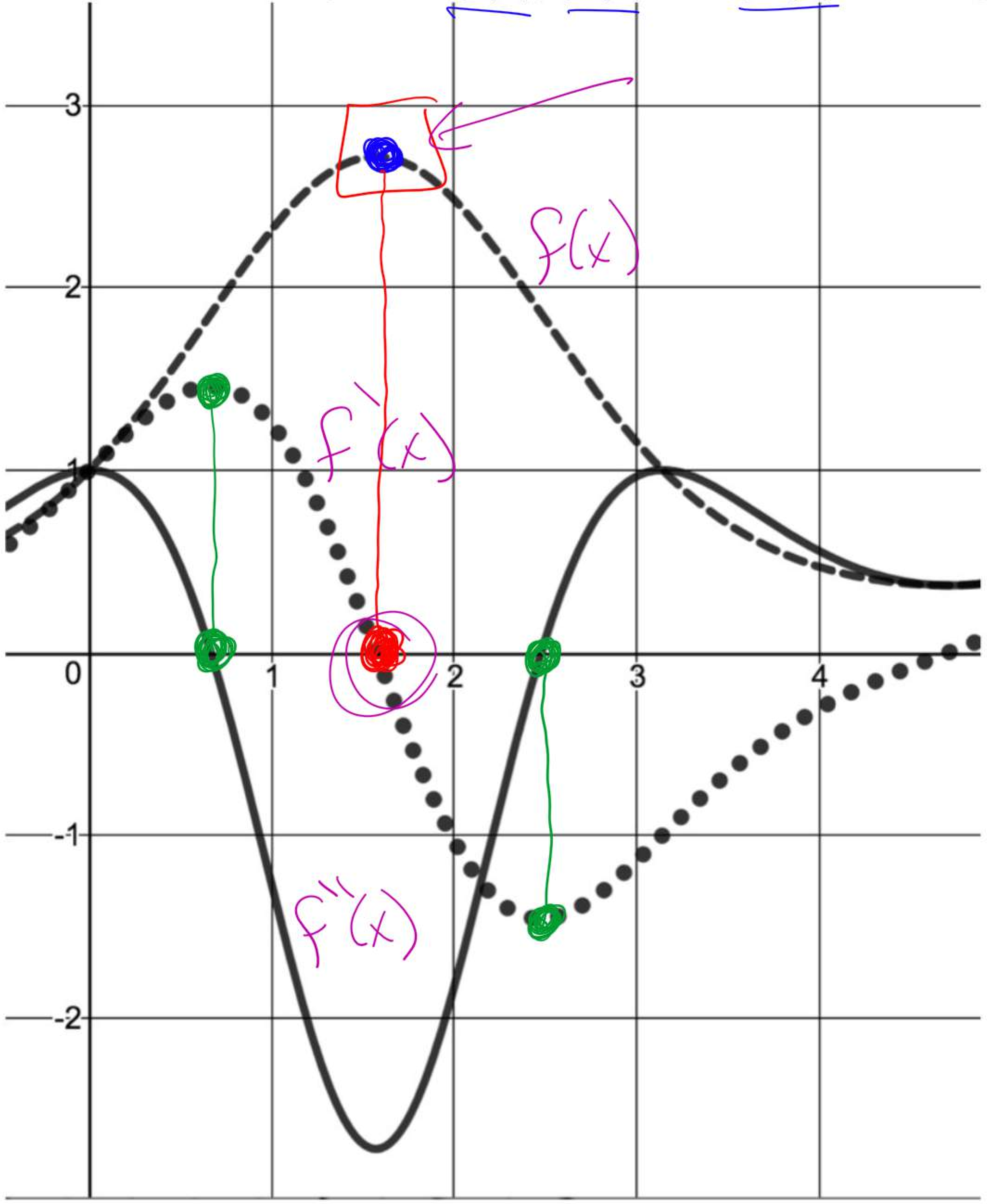
$$\boxed{h = \frac{1}{2}} \text{ C.V.}$$

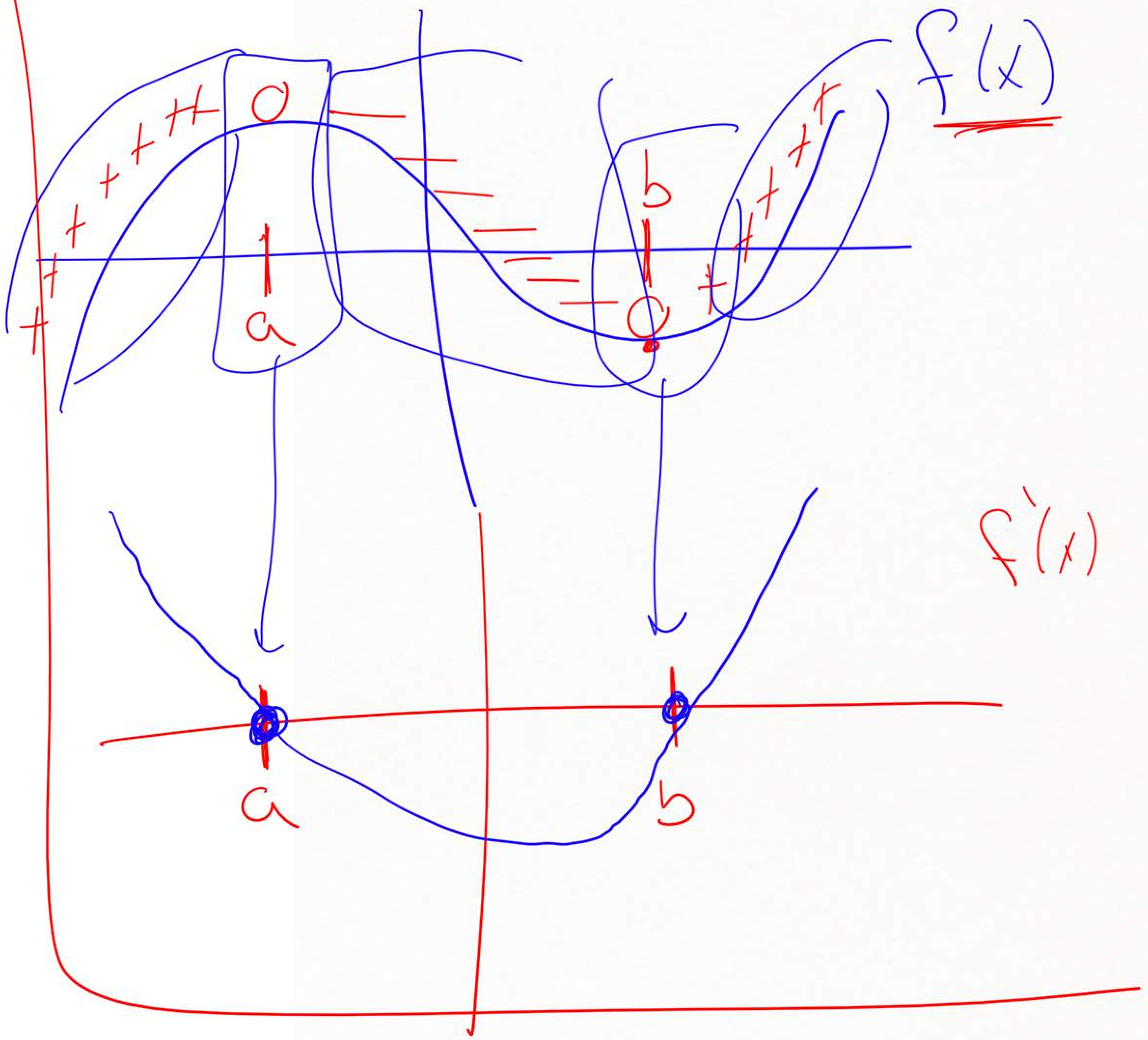


$$P'(0) = -1(0) + \frac{1}{2} = \frac{1}{2}$$

$$P'(1) = -1(1) + \frac{1}{2} = -\frac{1}{2}$$

7. (5 points) The following graph shows a function $f(x)$ and its first and second derivatives. Clearly label $f(x)$, $f'(x)$ and $f''(x)$ on the graph.



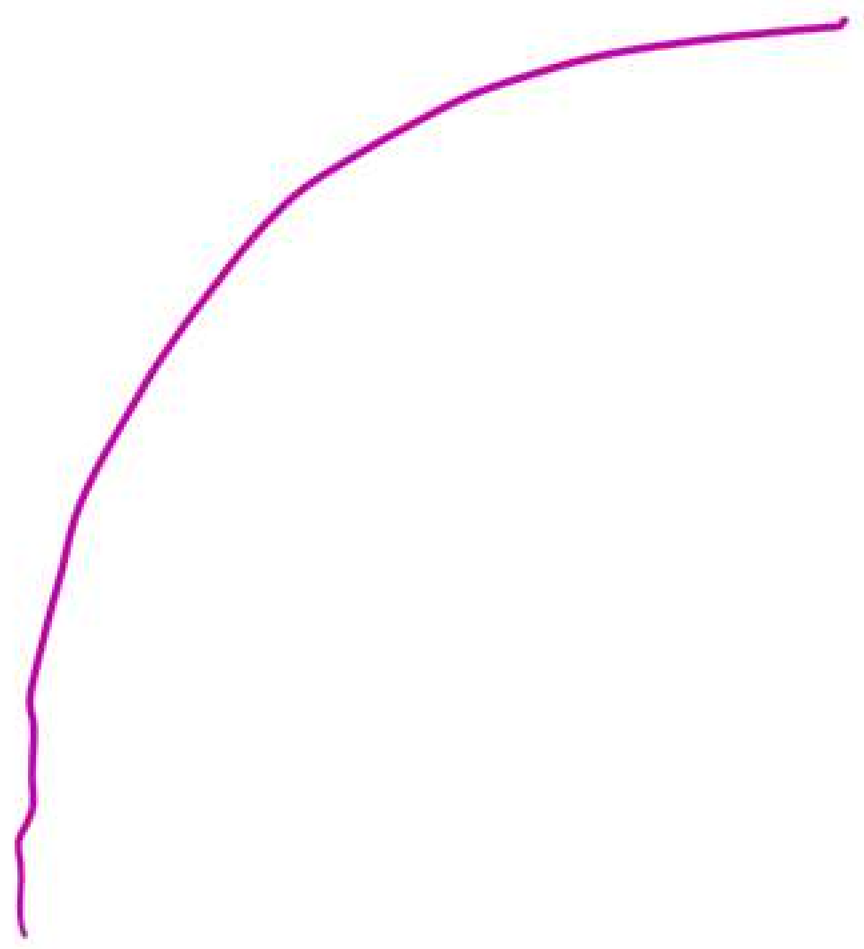


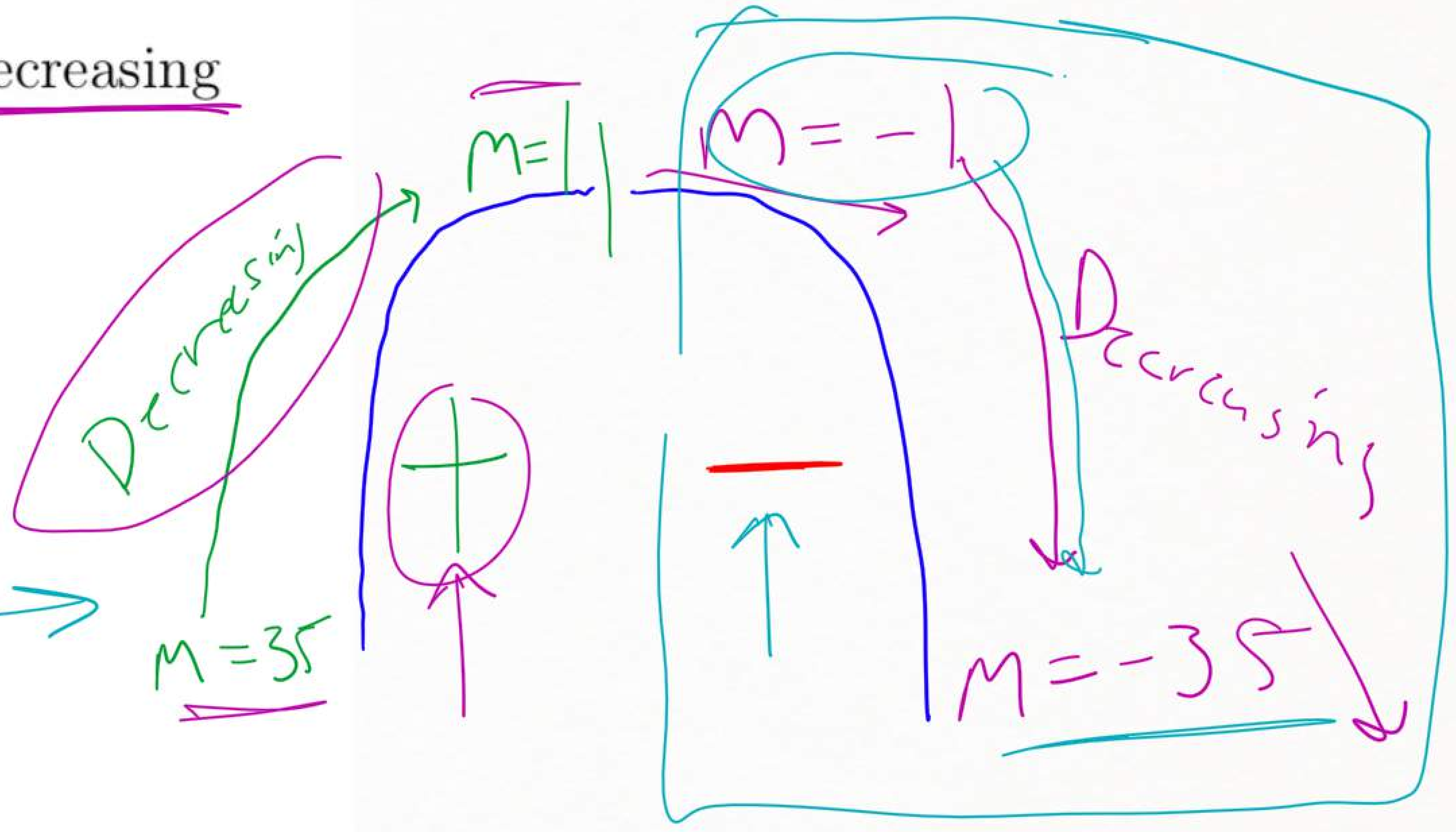
Dash → Dots → Line

f(x) f'(x) f''(x)

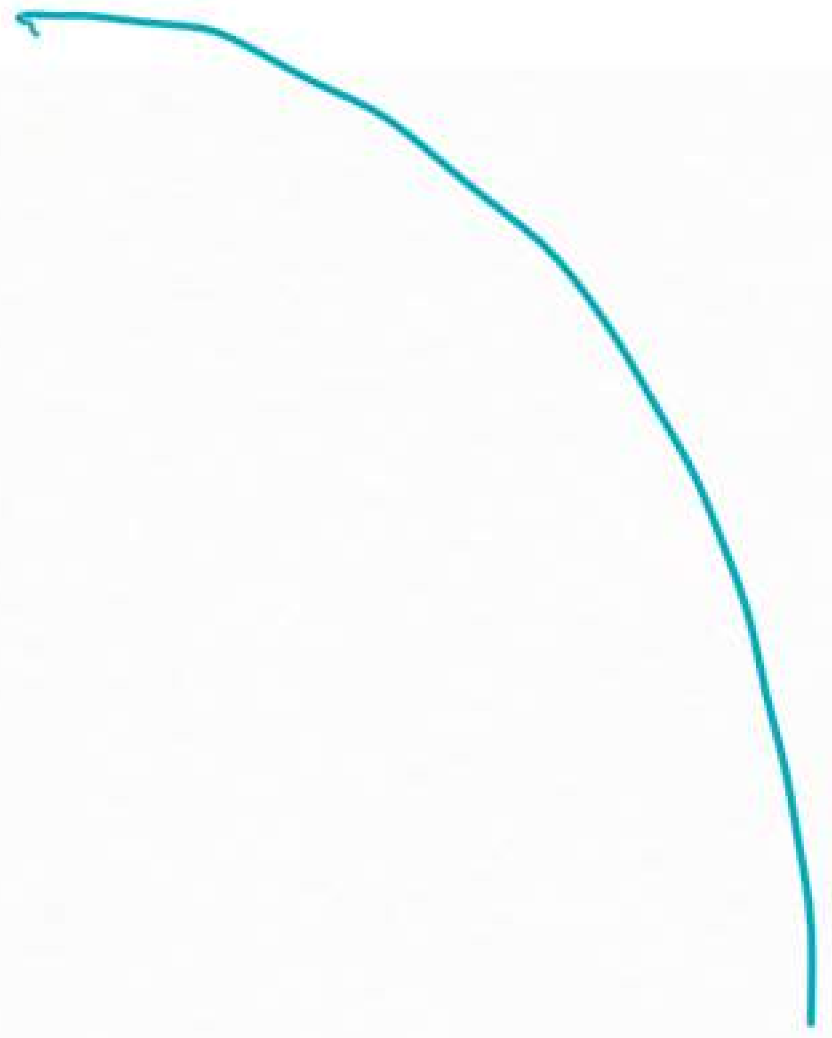
8. (8 points) Draw graphs of functions with the following properties. Label axis.

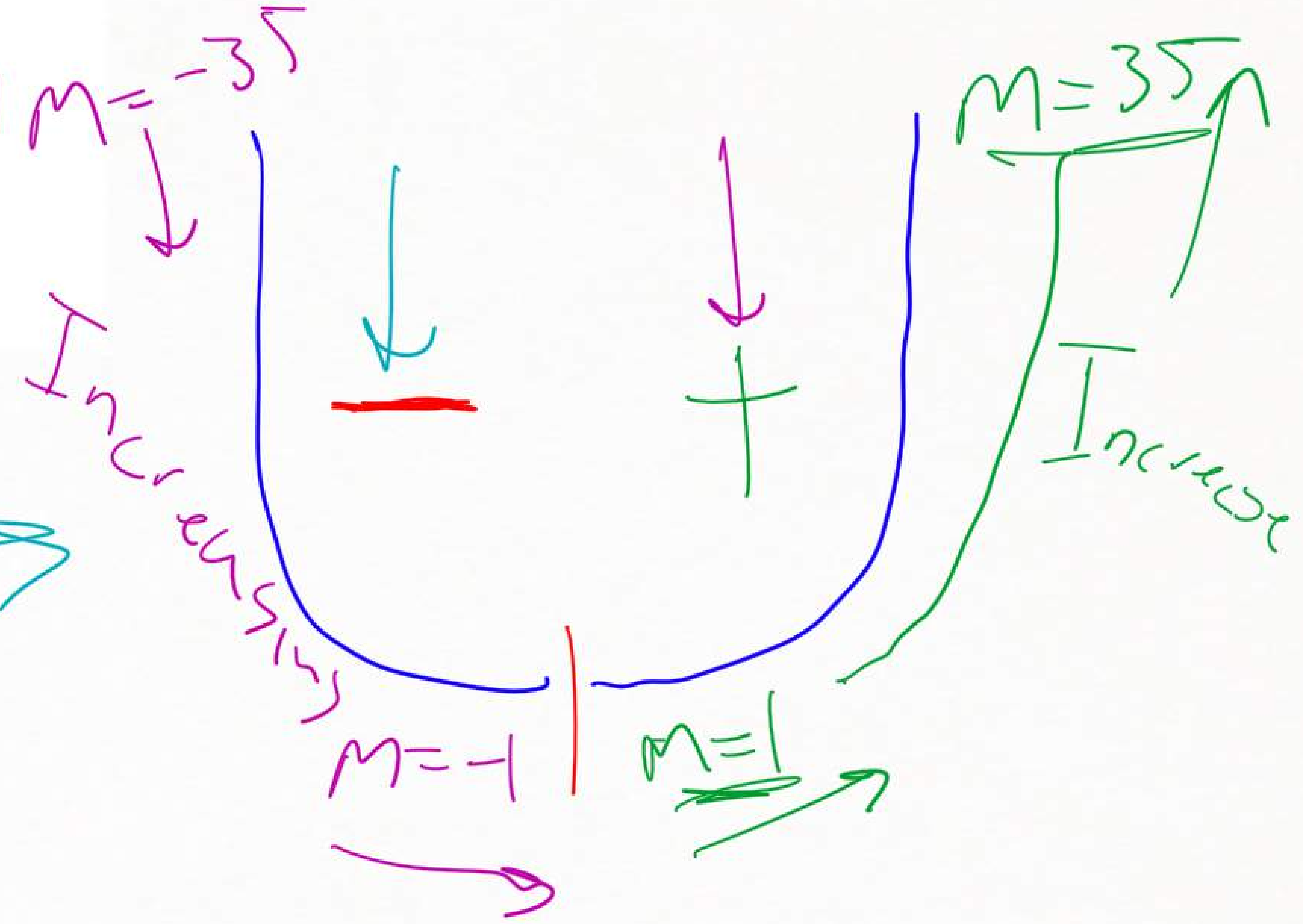
a) The function has a positive derivative for all values of x. The derivative is decreasing for all values of x.





b)The function has a negative derivative for all values of x. The derivative is decreasing for all values of x.





9. (6 points) The trachea contracts during a cough to increase the velocity of air. The velocity of air is modeled by

$$v(r) = c(r - 165)r^2$$

r is a positive variable that represents the radius of the trachea during a cough. c is a negative constant.

Find the value of r that maximizes the velocity, $v(r)$. Justify that for this value of r, $v(r)$ is maximized, using derivatives.

$$V(r) = c(r-165)r^2$$

$$V(r) = (cr - 165c)r^2$$

$$V(r) = cr^3 - 165cr^2$$

$$V'(r) = 3cr^2 - 330cr \; *$$

$$0 = 3cr^2 - 330cr \; *$$

$$0 = \boxed{3cr(r-110)}$$

$$\frac{3cr}{3c} = \frac{0}{3c}$$

$$\boxed{\cancel{r=0}}$$

$$\boxed{r - 110 = 0}$$

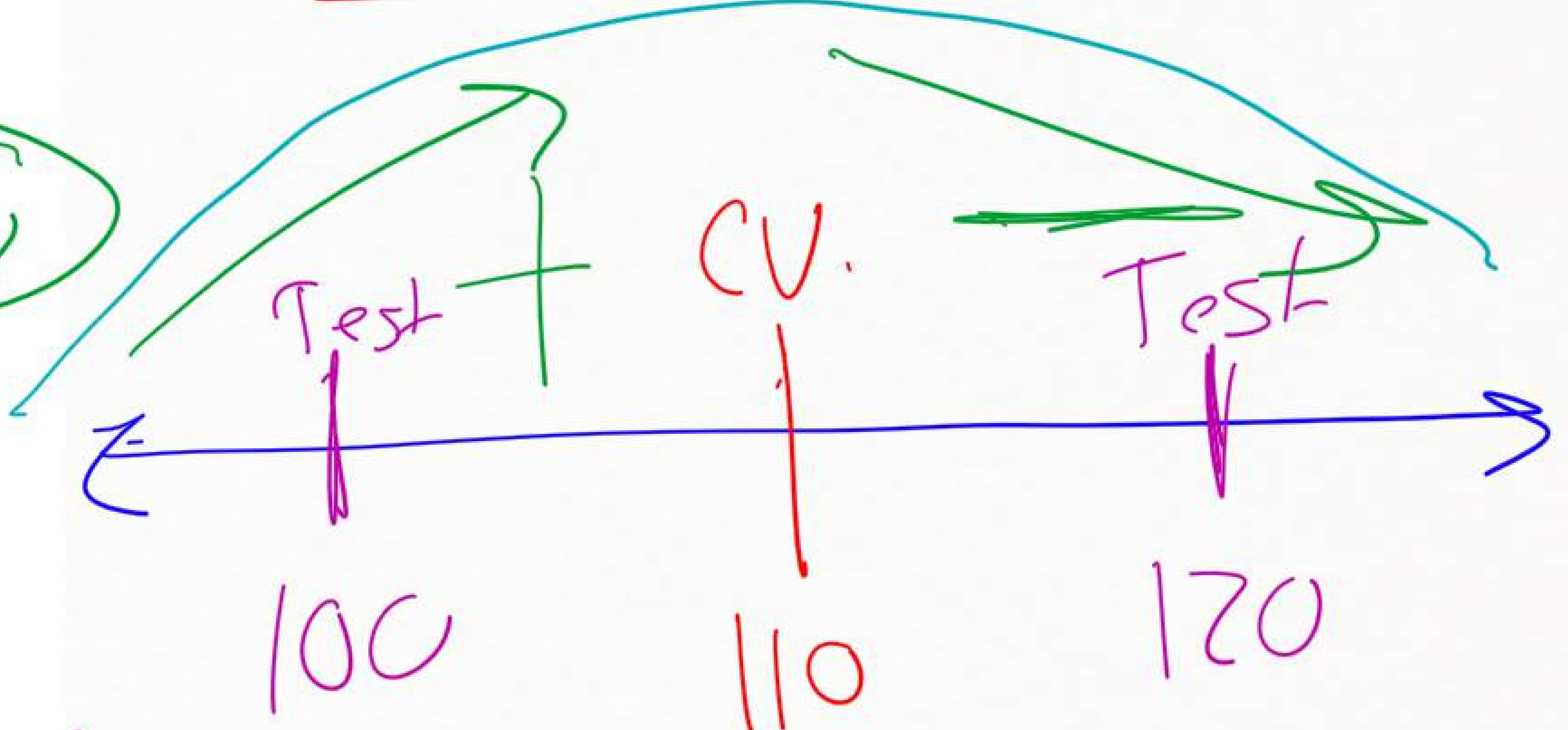
$$r - 110 = 0$$
$$+110 \quad +110$$
$$\boxed{r = 110} \; *$$

C.V.

$$V'(r) = 3cr(r-110)$$

$$\boxed{r = 110}$$

1st Derivative Test



$$V'(100) =$$

$$V'(100) = 3c(100)(100-110)$$

$$= \underbrace{300c}_{+\;-}(\underbrace{-10}_{-})$$

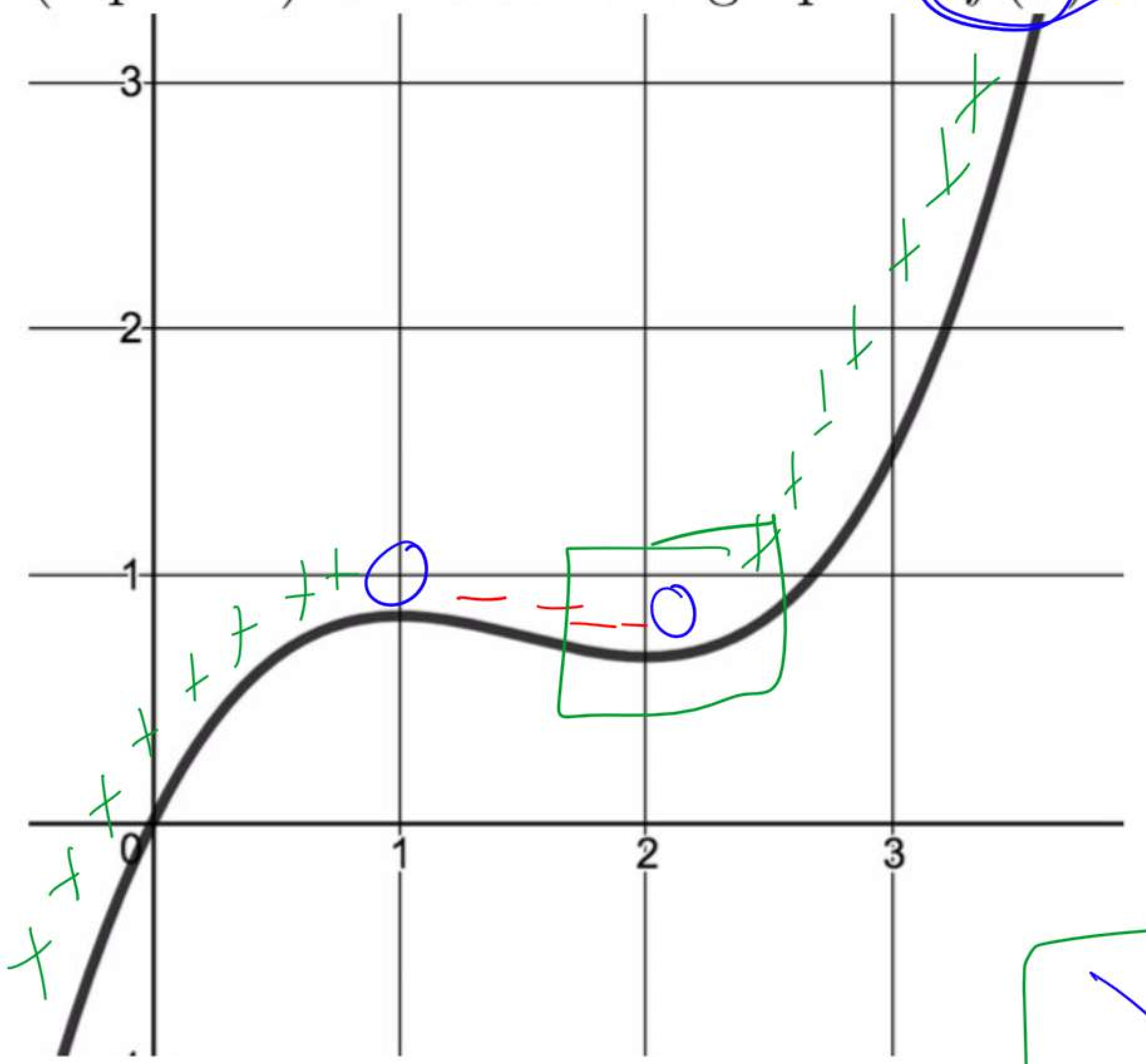
$$= \underbrace{+}\,\underbrace{-\cdot-}_{+}$$

$$= +$$

$$V'(120) =$$

$$V'(120) = \underbrace{\underbrace{3}_{+}\underbrace{c}_{-}}\,\underbrace{\underbrace{(120)}_{+}\underbrace{(120-110)}_{+}} = - \cdot + = -$$

10. (4 points) Consider the graph of $f(x)$ below.



The derivative, $f'(x)$, changes from negative to positive at:

$-$ $+$

a) $x = 1$

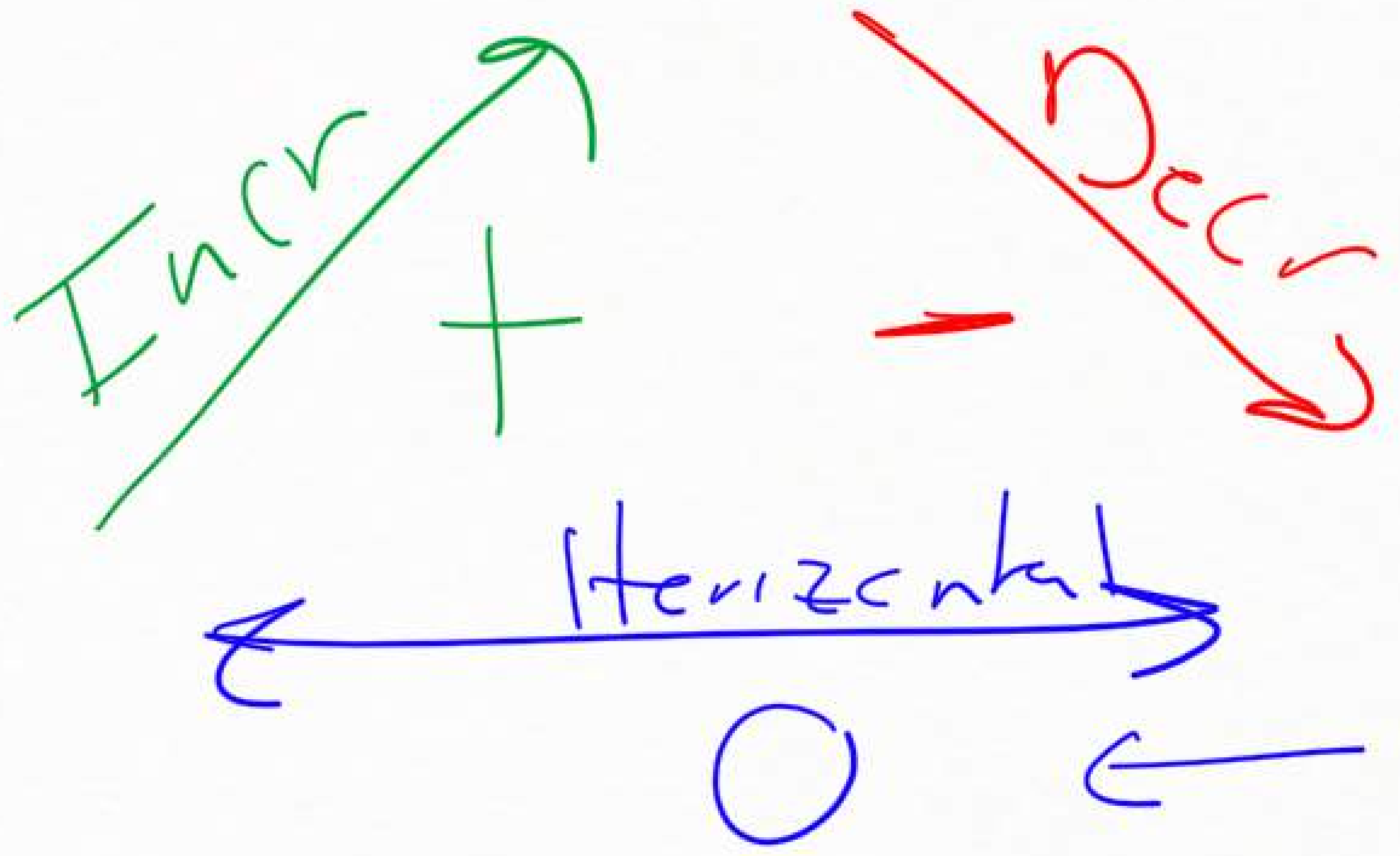
[✓ b) $x = 2$]

c) $x = 1.5$

d) $x = 0$

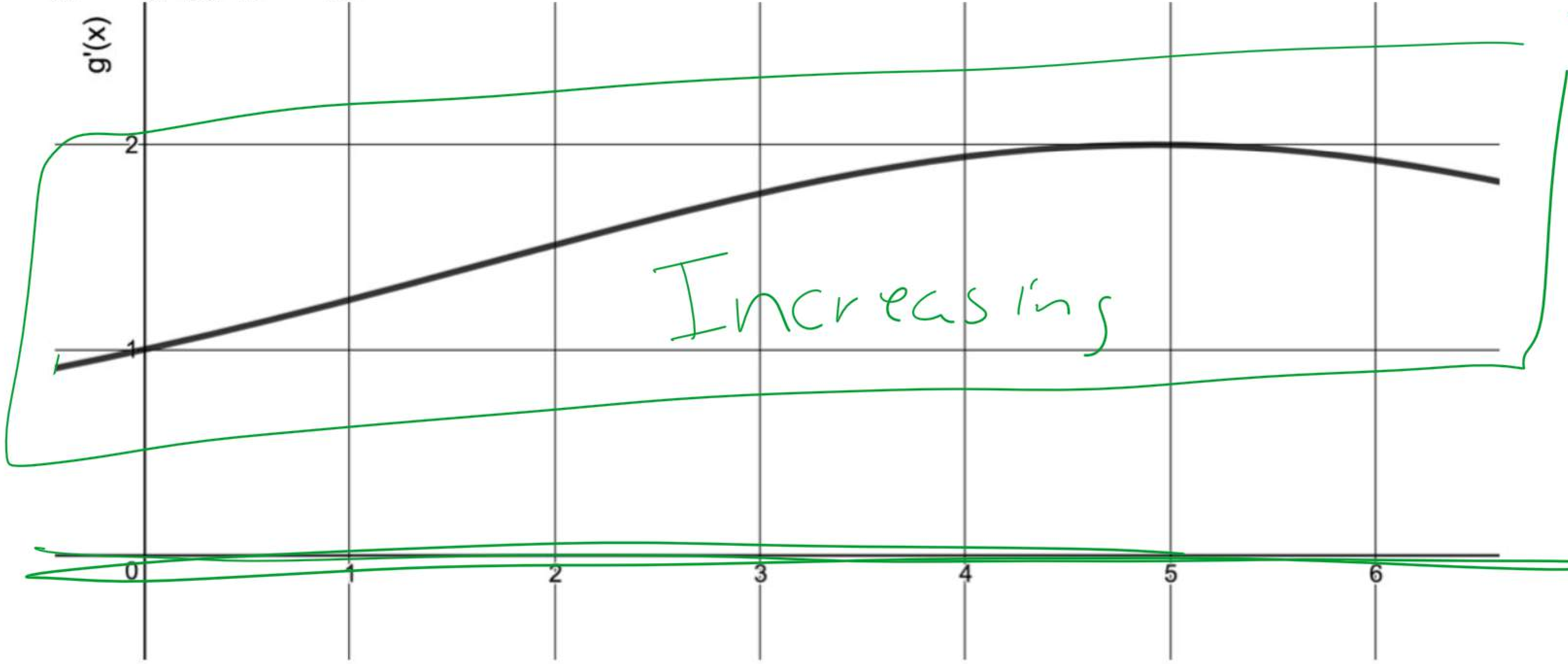
e) $x = 0.75$

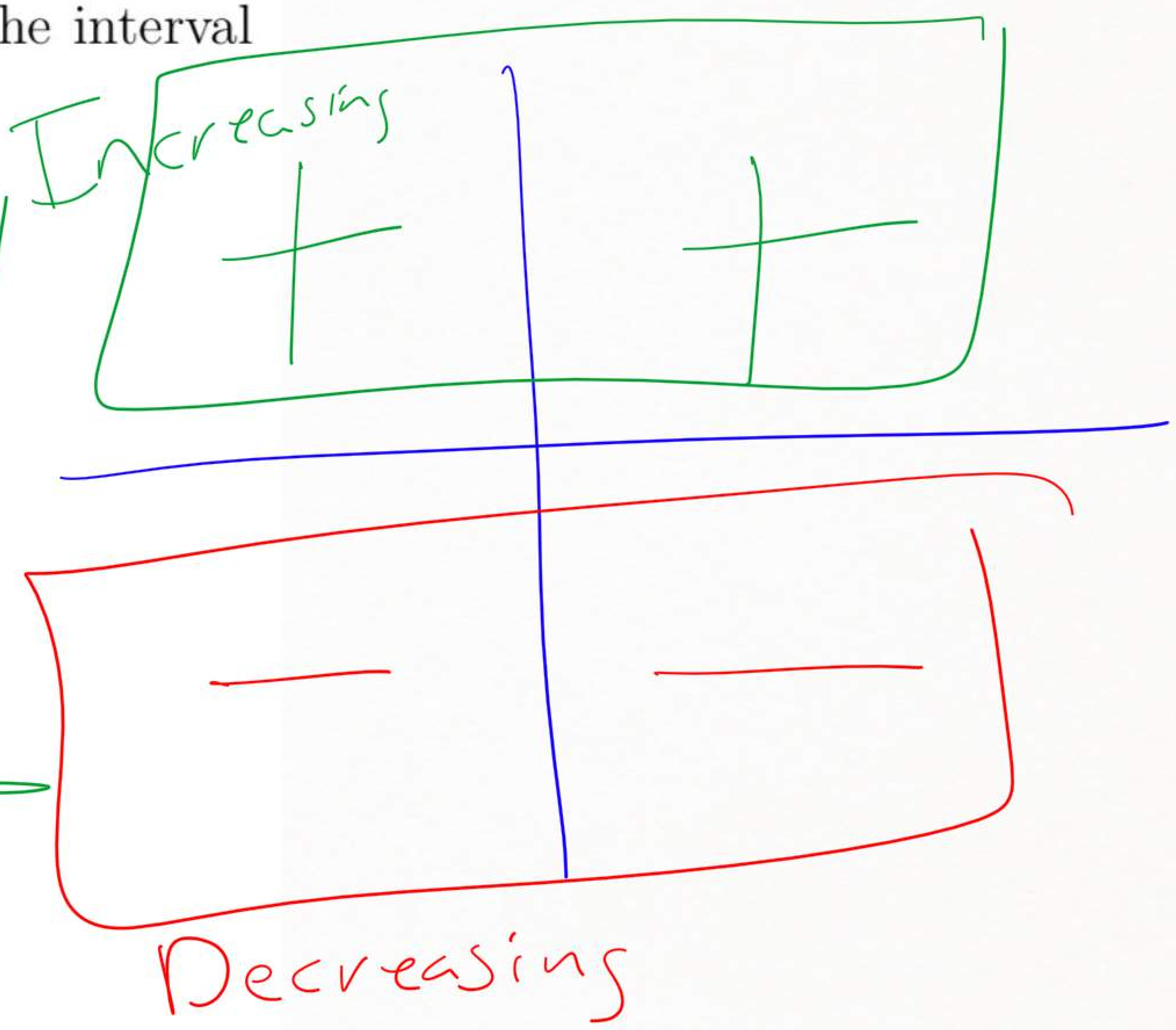
$f'(x) = $ slopes



← Critical Values

11. (4 points) Consider the graph of the **derivative**, $g'(x)$ below. Consider the interval $x = 0$ to $x = 6$.





As x increases, when does the value of the function $g(x)$ increase?

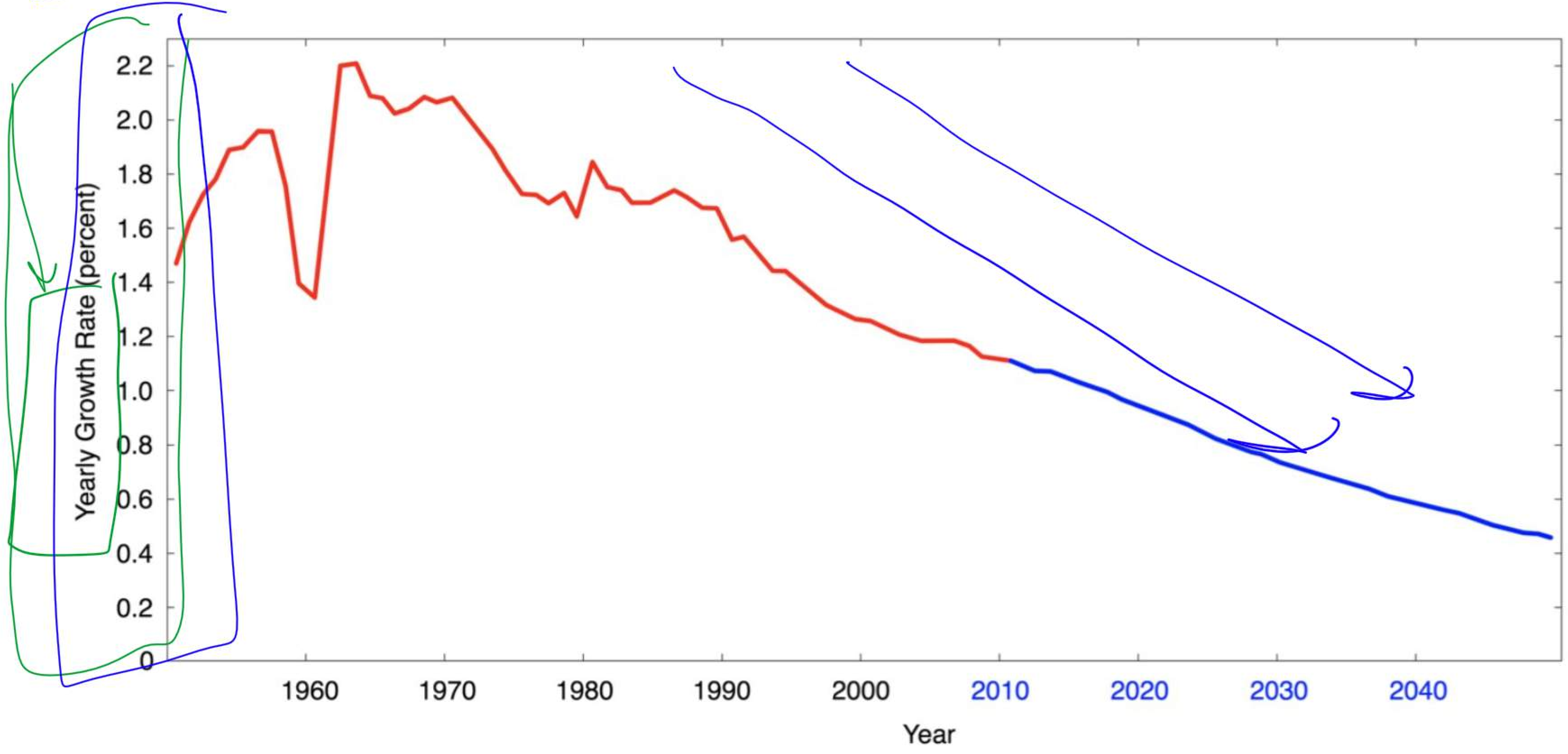
~~a) The function $g(x)$ does not increase on that interval.~~
[✓ b) The function $g(x)$ is increasing for all values of x on that interval.]
c) The function $g(x)$ is increasing on the interval $0 < x < 5$.
d) The function $g(x)$ is increasing on the interval $5 < x < 6$.
e) The function $g(x)$ is increasing on the interval $1 < x < 2$.

12. (4 points) Consider the following graph of the world population growth rate. A growth rate of 2% means that the population at the end of one year is 1.02 times as large as the population at the beginning of the year. The graph after 2018 is a model of what a group of scientists predicts will happen. Which of the following statements are consistent with the scientists' prediction?

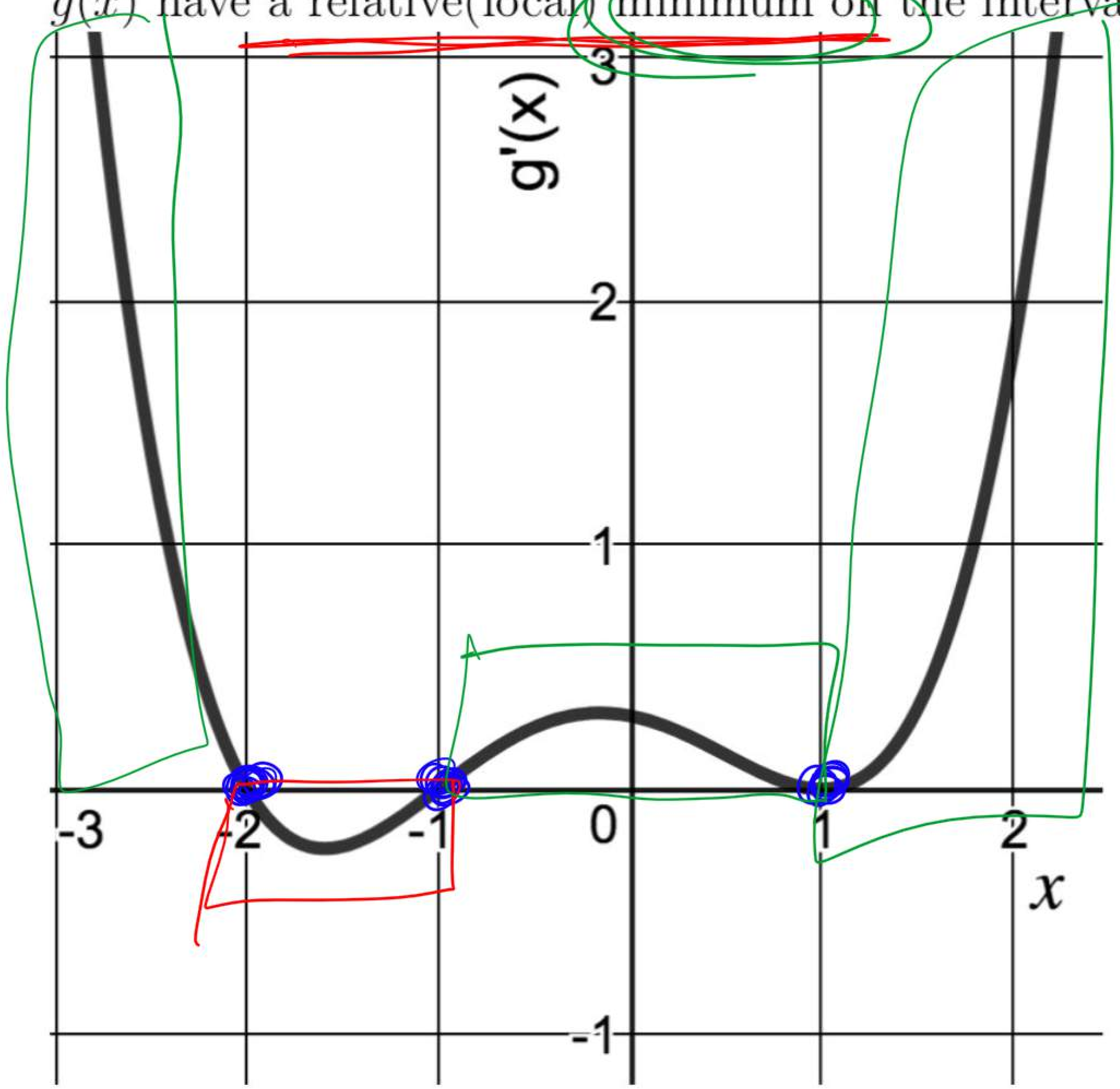
Growth Rate = positive

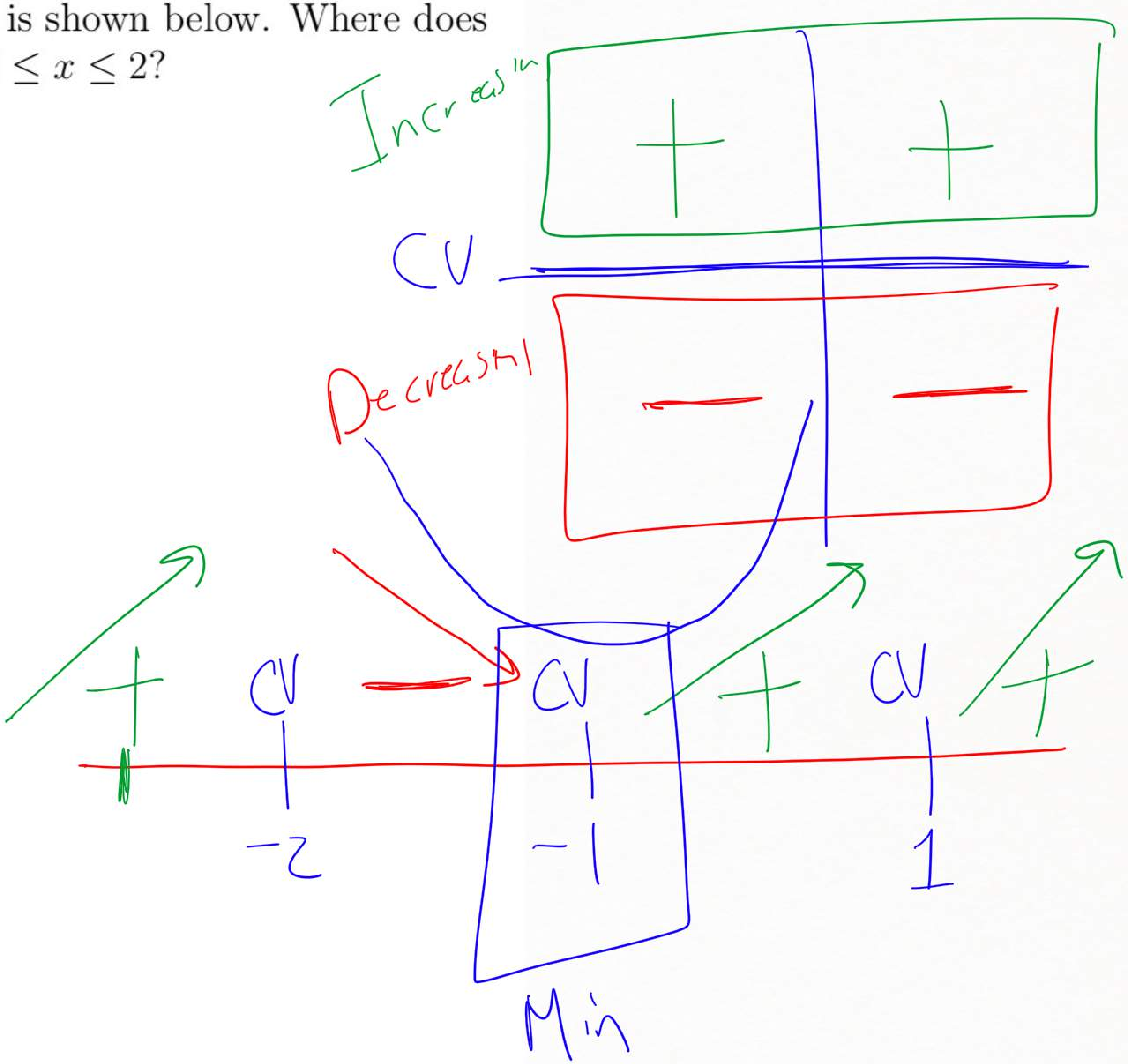
.4 positive amount
Growth

a) From 2018 to 2040 the world population will decrease.
b) From 2018 to 2040 the world population will increase.
c) From 2018 to 2040 the rate of change of the world population will decrease.
d) Both A and C.
[✓ e) Both B and C.]



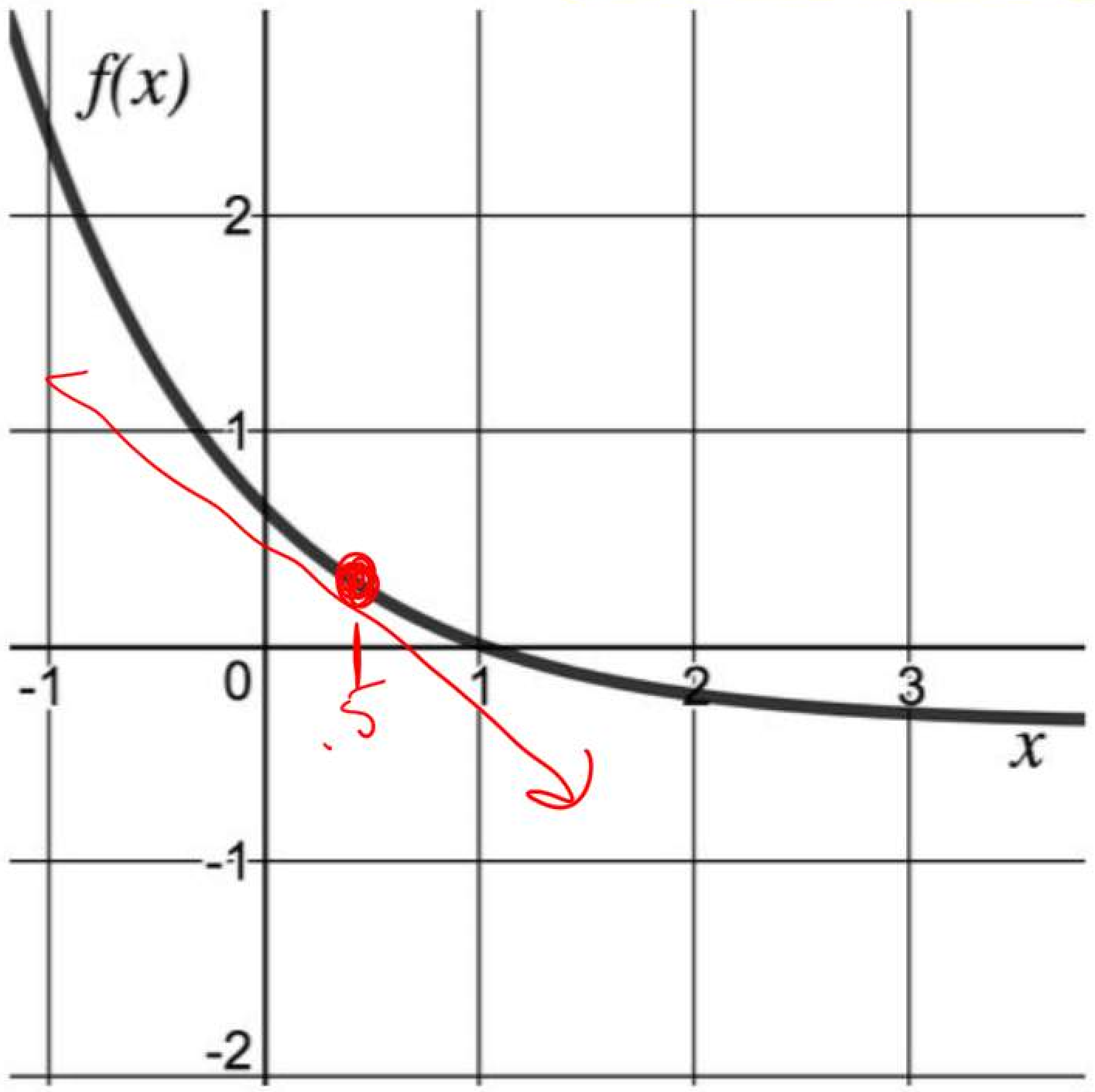
13. (4 points) The graph of the **derivative** of the function g is shown below. Where does $g(x)$ have a relative(local) minimum on the interval $-2.5 \le x \le 2$?





A) $x = 1$ and $x = -1.6$
B) $x = 1$
[✓ C) $x = -1$]
D) $x = -2.5$ and $x = -1$
E) $x = -2$

14. (4 points) The graph of $y = f(x)$ is given below.

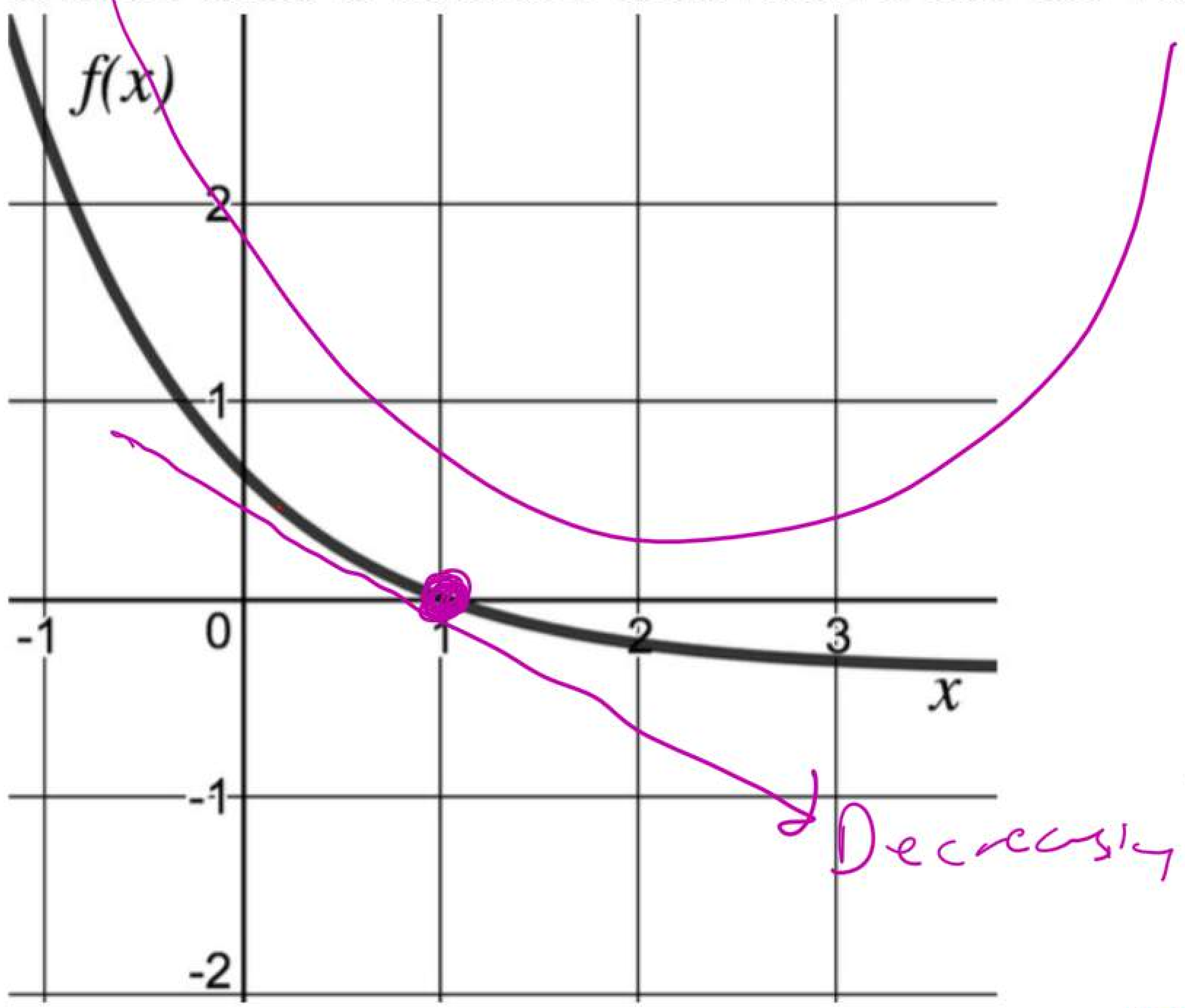


Decreasing = $f'(x)$ = Negative

The value of the **derivative**, $f'(x)$, at x = 0.5 is:

A) Positive
[✓ B) Negative]
C) Undefined
D) Zero
E) Not enough information.

15. (4 points) The graph of a function f is shown in the figure below. The function f has a first and a second derivative for all values of x. Which of the following is true?



A) $f(1) < f'(1) < f''(1)$

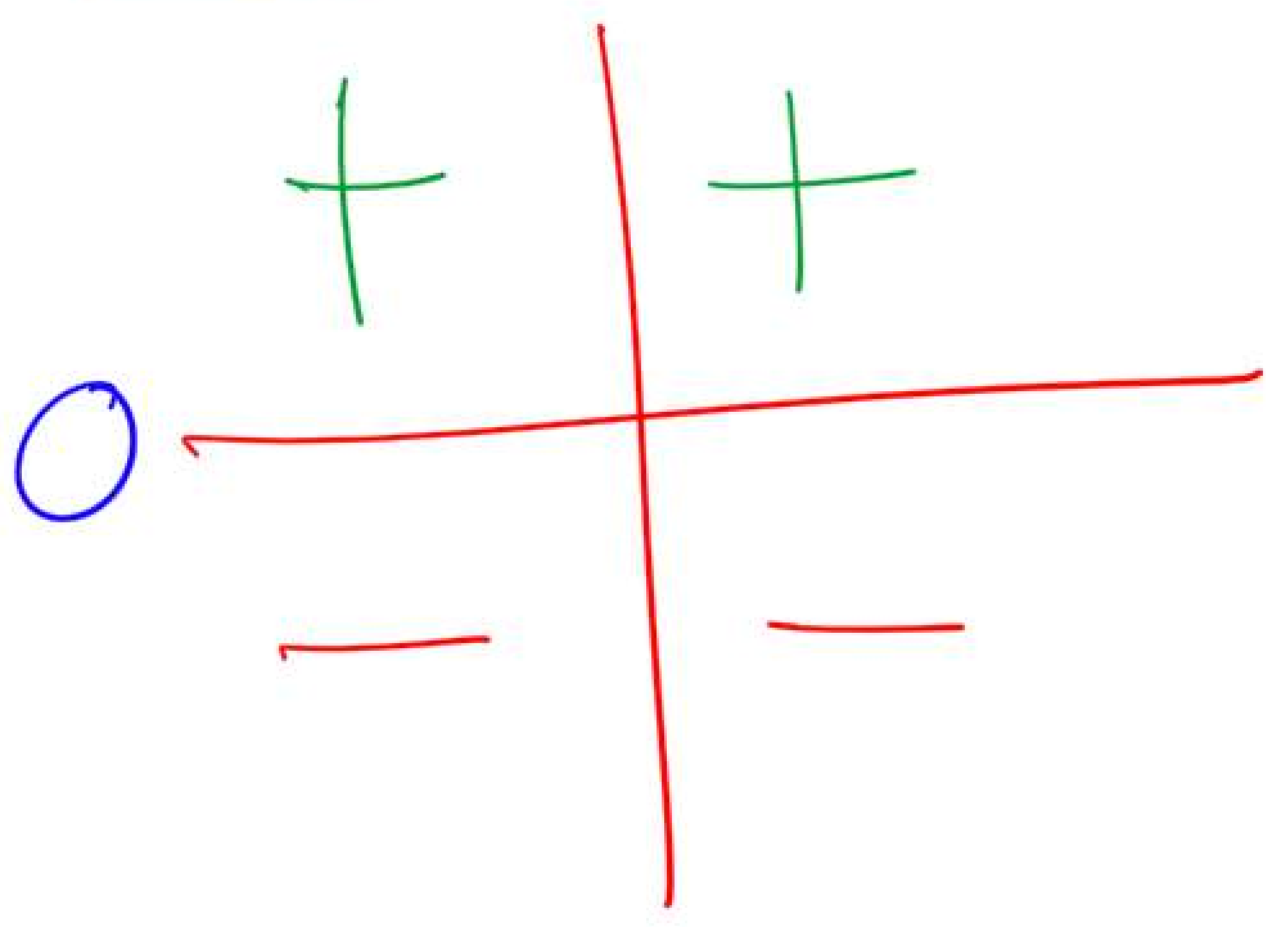
B) $f(1) < f''(1) < f'(1)$

[✓ C) $f'(1) < f(1) < f''(1)$]

D) $f''(1) < f(1) < f'(1)$

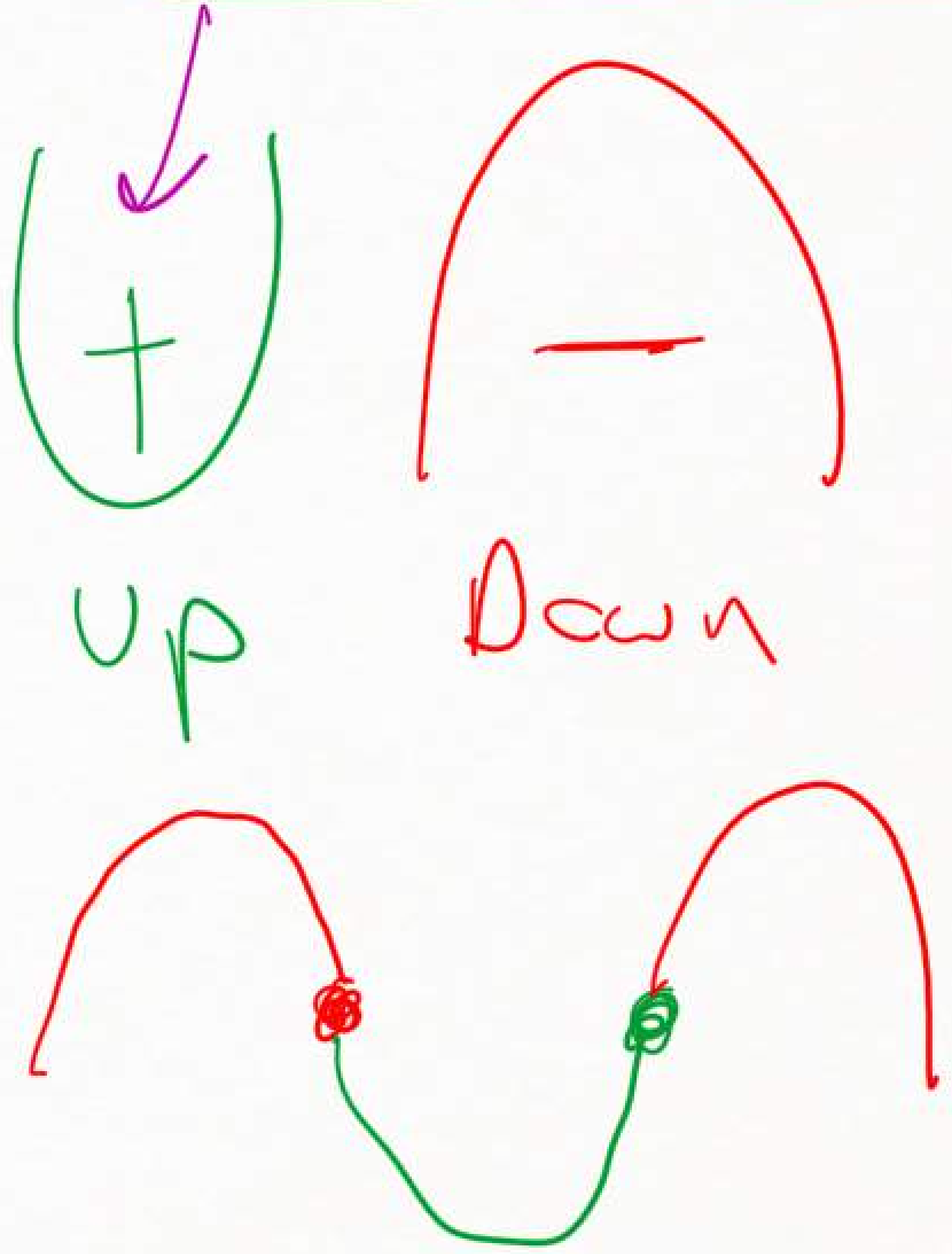
E) $f''(1) < f'(1) < f(1)$

$f(x) = $ y-value



$f'(x) = $ Slope

$f''(x) = $ Concavity



Inflection Pts

$f''(x) = 0$

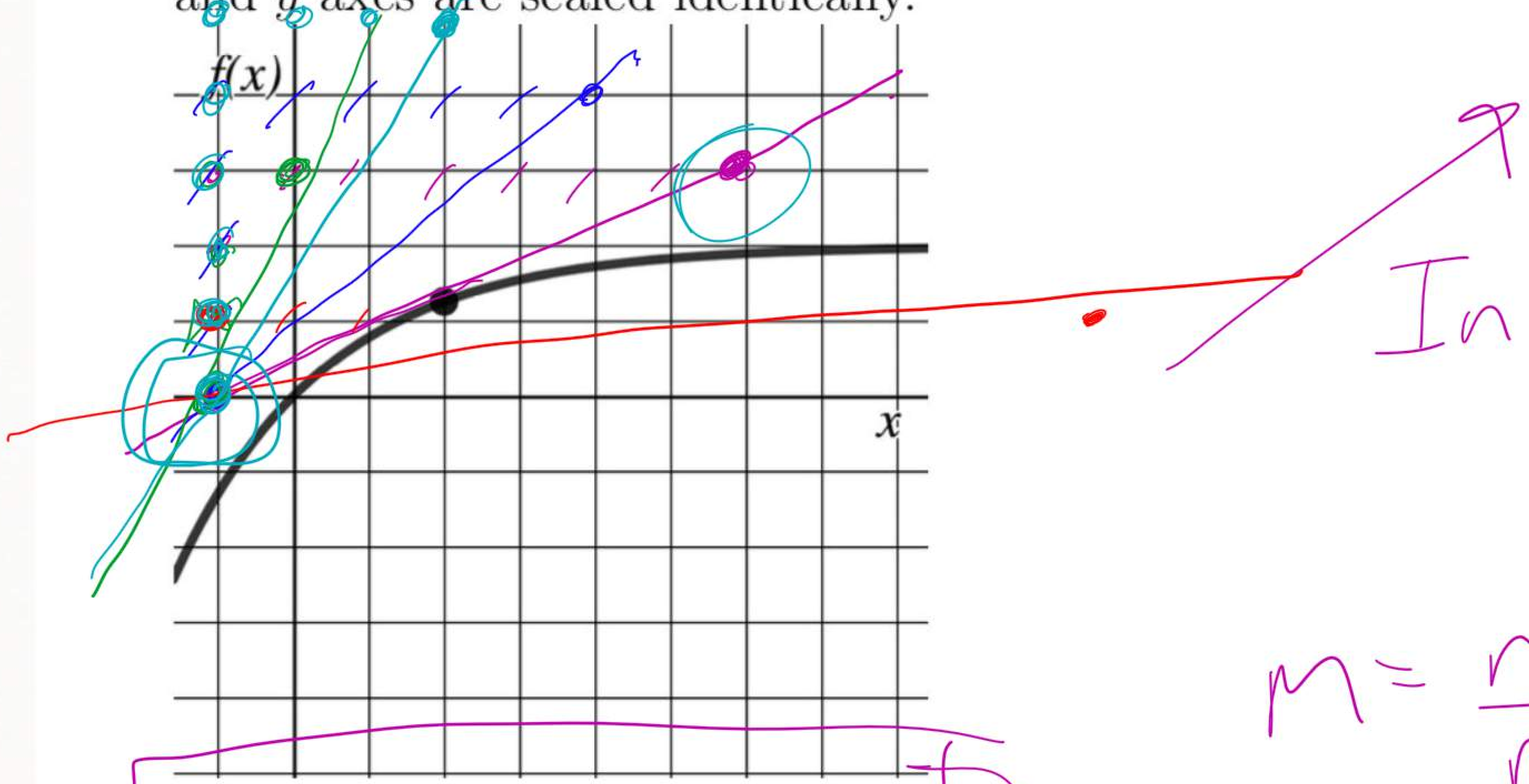
y-value → $f(1) = 0$

slope → $f'(1) = -$ Smallest

Concavity → $f''(1) = +$

$f'(1) < f(1) < f''(1)$

16. (4 points) Estimate the slope of the function at the point shown. See below. The x and y axes are scaled identically.



Inc positive

$m = \frac{rise}{run}$

$m = \frac{3}{7}$ ↑ up 3 → 7

[✓ A) The slope is approximately $\frac{3}{7}$]

B) The slope is approximately ~~$\frac{4}{5}$~~

C) The slope is approximately ~~3~~

D) The slope is approximately ~~$\frac{1}{20}$~~

E) The slope is approximately $\frac{5}{3}$

$m = \frac{3}{1}$ ↑ →

$m = \frac{4}{5}$ ↑ →

$m = \frac{1}{20}$ ↑ →

$m = \frac{5}{3}$ ↑ ↑ → →