Spring 2016 Final. Note: This years final will have a few MC problems about graphs of amount and rate functions like M2.

1. (14 points) Suppose that the population h_t of wild horses satisfies the discrete-time dynamical system

$$h_{t+1} = \frac{6}{5} h_t (k - h_t),$$

where $k > 0$ is a positive parameter.

(a) Find all equilibria. For what values of k is there more than one equilibrium that makes biological sense?

① Set h_{t+1} and h_t h^* $\quad h^* = \frac{6}{5} h^*(k - h^*)$

② Solve for h^*

$$h^* = \frac{6}{5} h^*(k - h^*)$$
$$-h^* \qquad -h^*$$

$$0 = \frac{6}{5} h^*(k - h^*) - h^*$$

$$0 = h^*\left(\frac{6}{5}(k - h^*) - 1\right) \leftarrow \boxed{h^* = 0} \text{ is a solution}$$

$$0 = \frac{6}{5}(k - h^*) - 1 \qquad \frac{5}{6} = k - h^* \qquad \boxed{h^* = k - \frac{5}{6}}$$

$$1 = \frac{6}{5}(k - h^*) \qquad \frac{5}{6} - k = -h^*$$

$h^* = 0$ makes biological sense.

$h^* = k - \frac{5}{6}$ has to be ≥ 0. Then we have:

$$k - \frac{5}{6} \geq 0$$

$$\boxed{k \geq \frac{5}{6}}$$

(b) For each equilibrium, use the Stability Theorem/Criterion to determine the values of k for which that equilibrium is stable. Show clearly how you are using the Stability Theorem/Criterion.

Remember: h^* is stable if $|f'(h^*)| < 1$ where

$$f'(x) = \left(\frac{6}{5} x (k - x)\right)' = \frac{6}{5}k - \frac{12}{5}x.$$

For $h^* = 0$ we have $f'(0) = \frac{6}{5}k$. Then $h^* = 0$ is stable when $\left|\frac{6}{5}k\right| < 1$.

$$-1 < \frac{6}{5}k < 1$$

$$\boxed{-\frac{5}{6} < k < \frac{5}{6}}$$

For $h^* = k - \frac{5}{6}$ we get

$$f'\left(k - \frac{5}{6}\right) = \frac{6}{5}k - \frac{12}{5}\left(k - \frac{5}{6}\right)$$
$$= \frac{6}{5}k - \frac{12}{5}k + \frac{12 \cdot 5}{6 \cdot 5}$$
$$= -\frac{6}{5}k + 2$$

Then $\left|-\frac{6}{5}k + 2\right| < 1$ gives

$$-1 < -\frac{6}{5}k + 2 < 1$$

$$-3 < -\frac{6}{5}k < -1$$

Multiply by -1 changes $<$ by $>$ and $>$ by $<$

$$3 > \frac{6}{5}k > 1$$

$$\boxed{\frac{15}{6} > k > \frac{5}{6}}$$

2. (14 points)

(a) Consider the function $f(x) = x^2 e^{-x}$.

i) Find *all* critical points of $f(x)$.

(productrule) $f'(x) = 2xe^{-x} + x^2(-e^{-x}) = 2xe^{-x} - x^2e^{-x}$

$= e^{-x}(2x - x^2)$

e^{-x} is never 0. Then the critical points are only when $2x - x^2 = 0$. Those are $\boxed{x=0}$ and $\boxed{x=2}$.

ii) Determine the global maximum and global minimum of $f(x)$ on the interval $[-1, 5]$. Justify your answer and show your work clearly for full credit.

The global max. and min are in the critical points or in the extremes of the interval.

We can check with a table:

Point p	0	2	-1	5
$f(p)$	0	0.5413	2.71828	0.16844
	minimum		maximum	

Then the global min is when $\boxed{x=0}$ and the global max is when $\boxed{x=-1}$

(b) Mary jumps from a diving board. Her height (in meters) above the water at time t (in seconds) is given by $h(t) = -5t^2 + 8t + 5$, and she jumps at time $t = 0$.

i) Find Mary's velocity $v(t)$ and acceleration $a(t)$ at time $t = 1$.

We know $v(t) = h'(t)$ and $a(t) = h''(t)$ Then

$v(t) = -10t + 8$

$a(t) = -10$

$\Rightarrow$ $\boxed{v(1) = -2 \quad a(1) = -10}$

ii) Use ***Calculus*** to determine the time at which Mary reaches her maximum height above the water. What is this height? Verify, ***using Calculus***, that this is a local maximum.

This means using 1st or 2nd derivative test.

① Find critical points:

$h'(t) = -10t + 8$

$0 = -10t + 8$

$-8 = -10t$

$\boxed{t = \frac{-8}{-10} = \frac{4}{5}}$

This height is

$h\left(\frac{4}{5}\right) = -5\left(\frac{4}{5}\right)^2 + 8\frac{4}{5} + 5$

$\boxed{h\left(\frac{4}{5}\right) = 8.2}$

② Show that the point $t = \frac{4}{5}$ is a maximum. You can use one of the following:

a) First derivative test:

0 $\quad \frac{4}{5} \quad$ 1

$+ \qquad -$

$h'(0) = 8 > 0 \quad +$

$h'(1) = -2 < 0 \quad -$

b) Second derivative test:

$h''\left(\frac{4}{5}\right) = -10 < 0$

Then the function is concave down.

3. (15 points) Evaluate the following definite and indefinite integrals. If necessary, use substitution. Show all of your work.

(a) $\int \frac{t^7-3t^2}{t^4} + 2\pi^2 \, dt$ $= \int t^3 - 3t^{-2} + 2\pi^2 \, dt = \boxed{\frac{1}{4}t^4 + 3t^{-1} + 2\pi^2 t + C}$

(b) $\int 2x^4 \sin(\pi + 3x^5) \, dx$ $=$ $\left| u = \pi + 3x^5, \; du = 15x^4 \, dx \Rightarrow dx = \frac{du}{15x^4} \right|$

$= \int \cancel{2x^4} \sin u \frac{du}{15\cancel{x^4}} = \frac{2}{15}\int \sin u \, du = -\frac{2}{15}\cos u + C = \boxed{\frac{-2}{15}\cos(\pi + 3x^5) + C}$

(c) $\int_0^2 \frac{t}{(t^2+4)^2} \, dt$ $=$ $\left| u = t^2 + 4, \; du = 2t\,dt \Rightarrow dt = \frac{du}{2t} \right.$

$\left. u\big|_{t=2} = 8, \; u\big|_{t=0} = 4 \right|$ Change the limits using $u = t^2 + 4$

$= \int_4^8 \frac{\cancel{t}}{u^2} \cdot \frac{du}{2\cancel{t}} = \frac{1}{2}\int_4^8 \frac{du}{u^2} = -\frac{1}{2}\frac{1}{u}\Big|_4^8 = -\frac{1}{2}\left(\frac{1}{8} - \frac{1}{4}\right) = \boxed{\frac{1}{16}}$

(d) Use integration by parts to evaluate $\int -2x \sin(9x) \, dx$. $=$

$\left| u = 2x, \; dV = -\sin(9x)\,dx \Rightarrow V = \frac{1}{9}\cos(9x) \right.$
$\left. du = 2\,dx \right|$

$= \int u\,dV = uV - \int V\,du = \frac{2}{9}x\cos(9x) - \int \frac{1}{9}\cos(9x)\,2\,dx =$

$= \boxed{\frac{2}{9}x\cos(9x) - \frac{2}{81}\sin(9x)}$

4. (14 points) Suppose that a bacterium is absorbing lactose from its environment. At time $t = 0$, there is 0.2 mol of lactose in the bacterium, and lactose enters the bacterium at a rate of $0.1 \sin^2(t) \frac{\text{mol}}{\text{hour}}$

(a) Let $L(t)$ represent the amount (mol) of lactose in the bacterium at time t (hours). Write a pure-time differential equation and an initial condition for the situation described above.

$$\boxed{\frac{dL}{dt} = 0.1 \cdot \sin^2(t)}, \quad \boxed{L(0) = 0.2}$$

(b) Apply Euler's Method with $\Delta t = 0.5$ to estimate the amount of lactose in the bacterium at time $t = 1.5$. Show your work clearly. Give your answer to three decimal places. (Recall the formula $\hat{L}_{\text{next}} = \hat{L}_{\text{current}} + \frac{dL}{dt}\Delta t$, or $\hat{L}(t + \Delta t) = \hat{L}(t) + L'(t)\Delta t$).

we didnt call using $\Delta y = m \Delta x$ formula to approximate amounts from rates Euler's method. Now you know Euler invented method we used.

$$L(1.5) = L(0) + L'(0)\Delta t + L'(0.5)\Delta t + L'(1) \cdot \Delta t$$

$$\boxed{\Delta t = 0.5}$$

$$\Delta t \cdot L'(0) = 0.1 \cdot \sin^2(0) \cdot 0.5 = 0$$

$$\Delta t \cdot L'(0.5) = 0.1 \cdot \sin^2(0.5) \cdot 0.5 = 0.0115$$

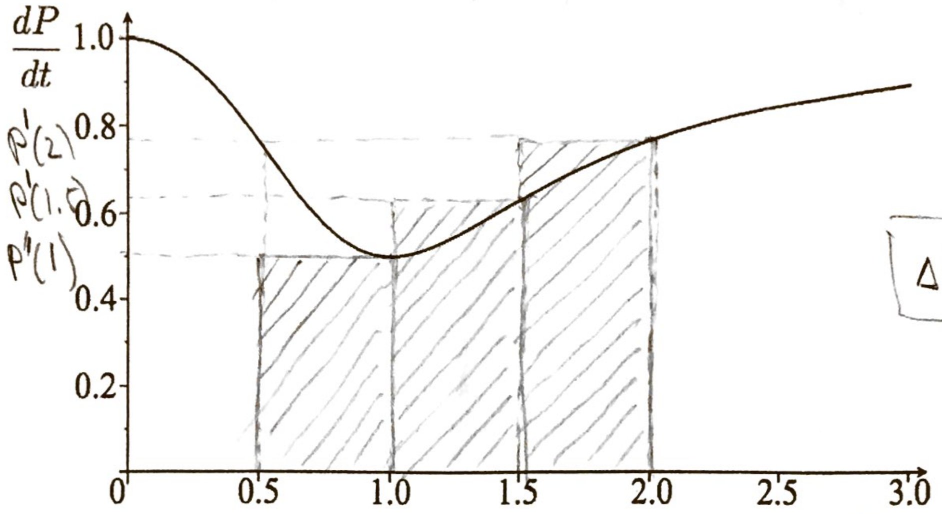
$$\Delta t \cdot L'(1) = 0.1 \cdot \sin^2(1) \cdot 0.5 = 0.0354$$

$$\Rightarrow L(1.5) = 0.2 + 0 + 0.0115 + 0.0354 = \boxed{0.247} \quad (0.246896)$$

5. (14 points) A western wood pewee slows down a bit to catch a fly, and then increases its speed again as it flies on. Denoting the position (in meters) of the pewee at time t (in seconds) by $P(t)$, suppose that the pewee's velocity is given by

$$\frac{dP}{dt} = 1 - \frac{t^2}{1+t^4}.$$

(a) Estimate the total change in $P(t)$ between times $t = 0.5$ and $t = 2$ using a right-hand Riemann Sum with $\Delta t = 0.5$. Draw your rectangles or step functions on the graph below.



$\boxed{\Delta t = 0.5}$

$\Delta P = P'(2)\cdot\Delta t + P'(1.5)\Delta t + P'(1)\cdot\Delta t = \Delta t\left[P'(2) + P'(1.5) + P'(1)\right]$

$P'(2) = 1 - \frac{4}{1+16} = \frac{17-4}{17} = \frac{13}{17}$; $P'(1.5) = 1 - \frac{(1.5)^2}{1+(1.5)^4} = 0.6288$

$P'(1) = 1 - \frac{1}{1+1} = \frac{1}{2}$

$\Rightarrow \Delta P = 0.5\left[\frac{13}{17} + 0.6288 + \frac{1}{2}\right] = \boxed{0.947}$ (0.94678544)

(b) Find the average value of the function $\frac{4t}{3+t^2}$ between $t = 0$ and $t = 2$. The average value is $\hat{f}$

$\int_a^b f(x)\,dx = \hat{f}\cdot(b-a) \Rightarrow \boxed{\hat{f} = \frac{\int_a^b f(x)\,dx}{b-a}}$

$\int_0^2 \frac{4t}{3+t^2}\,dt =$ $\left| \begin{array}{l} u = 3+t^2,\ du = 2t\,dt \Rightarrow dt = \frac{du}{2t} \\ u|_{t=2} = 7,\ u|_{t=0} = 3 \end{array} \right|$

$= \int_3^7 \frac{4t}{u}\frac{du}{2t} = 2\int_3^7 \frac{1}{u}\,du = \underline{2(\ln 7 - \ln 3)}$ + $\boxed{(b-a) = 2}$

$\Rightarrow \boxed{\text{Answer}} = \frac{2(\ln 7 - \ln 3)}{2} = \boxed{\ln 7 - \ln 3} = \boxed{0.847}$

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6. (15 points)

(a) Andy's horse starts with a concentration of medicine in her bloodstream equal to 3 milligrams per liter (mg/L). Each day, the horse uses up 23% of the medicine in her bloodstream. However, at the end of each day the vet gives her enough medication to increase the concentration of medicine in the bloodstream by 2 mg/L. Let M_t = concentration of medicine on day t, and write down a discrete-time dynamical system, together with an initial condition, that describes this situation.

$$M_{t+1} = 0.77 \cdot M_t + 2, \quad M_0 = 3$$

(b) Let $V(t)$ = the volume (in liters) of blood at time t (in seconds) in Dean's liver. Suppose that

$$\frac{dV}{dt} = 0.2\cos(\pi t - \pi/2).$$

i. Use a definite integral to determine the total change in the volume of blood in Dean's liver between times $t = 1$ and $t = 4$.

$$\Delta V = \int_1^4 \frac{dV}{dt}\,dt = \int_1^4 0.2\cos(\pi t - \pi/2) = $$

$u = \pi t - \frac{\pi}{2}$, $du = \pi\, dt \Rightarrow$ $dt = \frac{du}{\pi}$, $u|_4 = \frac{7}{2}\pi$, $u|_1 = \frac{\pi}{2}$

$$= \int_{\frac{\pi}{2}}^{\frac{7\pi}{2}} 0.2\cos u \cdot \frac{du}{\pi} = \frac{0.2}{\pi} \sin u \Big|_{\frac{\pi}{2}}^{\frac{7\pi}{2}} = \frac{0.2}{\pi}(-1-1) = \boxed{-\frac{0.4}{\pi}}$$

ii. Determine $V(t)$ if $V(0) = 0.47$. (That is, find a solution to the differential equation $\frac{dV}{dt} = 0.2\cos(\pi t - \pi/2)$ with initial condition $V(0) = 0.47$.)

$$V(t) = \int 0.2\cos\left(\pi t - \frac{\pi}{2}\right) dt = \boxed{\frac{0.2}{\pi}\sin\left(\pi t - \frac{\pi}{2}\right) + C} \quad (\text{from (i)})$$

$$V(0) = \frac{0.2}{\pi}\sin\left(-\frac{\pi}{2}\right) + C = -\frac{0.2}{\pi} + C = 0.47 \Rightarrow$$

$$C = 0.47 + \frac{0.2}{\pi} = \boxed{0.534} \; (0.5336619)$$

$$\Rightarrow \boxed{V = \frac{0.2}{\pi}\sin\left(\pi t - \pi/2\right) + 0.534}$$

7. (14 points)

(a) A population p_t of puffins on an island obeys the discrete-time dynamical system

$$p_{t+1} = 1.12p_t.$$

i. Write down the solution to this discrete-time dynamical system if $p_0 = 1234$.

$$P_t = (1.12)^t P_0 = (1.12)^t \cdot 1234$$

ii. If $p_0 = 1234$, at what time will the population reach size 2000?

$$2000 = (1.12)^t \cdot 1234 = e^{t \cdot \ln(1.12)} \cdot 1234 \Rightarrow$$

$$e^{t \cdot \ln(1.12)} = \frac{2000}{1234} \Rightarrow \boxed{t = \frac{1}{\ln(1.12)} \cdot \ln\left(\frac{2000}{1234}\right) \approx 4.2609}$$

(b) The density ρ of a very thin rod (measured in grams/cm) varies according to

$$\rho(x) = 4xe^{-2x^2},$$

where x marks a location along the rod, and $x = 0$ at one end of the rod. What is the total mass of the rod if it is 5 cm long? Give units in your answer.

$$M = \int_0^5 \rho(x)\,dx = \int_0^5 4xe^{-2x^2}\,dx \; \textcircled{=}$$

$$\boxed{u = -2x^2,\; du = -4x\,dx \Rightarrow dx = \frac{du}{-4x},\; u|_0 = 0,\; u|_5 = -50}$$

$$\textcircled{=} \int_0^{-50} 4xe^u \cdot \frac{du}{-4x} = -e^u \Big|_0^{-50} =$$

$$= \boxed{-e^{-50} + 1} \quad (\approx 1)$$

(c) Suppose that a population $p(t)$ of porcupines satisfies the differential equation

$$\frac{dp}{dt} = 1.11p(10 - p).$$

i. Find all equilibria of the differential equation.

set $\frac{dP}{dt} = 0$, and solve for P.

$$p^* = 0, \; p^* = 10$$

ii. Write down an initial condition for which the population will increase in time (at least initially).

$p(0) =$ 5 *

* You can pick your favorite number in $(0,10)$ (explanation below)

The population increase when: $\frac{dp}{dt} > 0 \Rightarrow p(10-p) > 0 \Rightarrow \boxed{p \in (0,10)}$