Math 155 **Exam 2** **Spring 2018**

NAME: Key

SECTION: ____________ TIME: ____________

INSTRUCTOR: ____________

About 70% of Fall, 2018 will be very similar. 30% will use same concepts to solve problems from homework on webwork.

Instructions: The exam is closed book and closed notes. You may use an approved calculator, but be sure to *clearly* show your work on each problem for full credit. Work that is crossed out or erased will not be graded. If you need scratch paper ask for some from the proctor. Turn in any scratch paper that you use during the exam. You will have one hour and 50 minutes to work on the exam.

Problem	Points	Score
1	12	
2	9	
3	10	
4	9	
5	9	
6	9	
7	6	
8	5	
9	6	
10	5	
11	5	
12	5	
13	5	
14	5	
Bonus	3	
Total	100	

CONFIDENTIALITY PLEDGE

I agree that I will not share any information, either specific or general, about the problems on this examination with any other person until the exams have been returned to us in class.

____
(Signature)

1. (12 points) Compute the derivatives of the following functions. You do NOT need to simplify your answer. Use parentheses to indicate multiplication where appropriate, and make sure that your notation is correct.

(a) $f(x) = \sqrt{x} + e^{3x^2}$

chain rule on second part

$f(x) = x^{1/2} + e^{3x^2}$

$\boxed{f'(x) = \frac{1}{2}x^{-1/2} + (e^{3x^2})(6x)}$

(b) $f(t) = \ln(\sin(t))$

chain rule

$\boxed{f'(t) = \frac{1}{\sin(t)} \cdot \cos(t)}$

(c) $p(x) = (\overset{u}{e^x + 1})(\overset{v}{x^2 - 2x + 1})$

product rule

$p'(x) = udv + vdu$

$\boxed{p'(x) = (e^x+1)(2x-2) + (x^2-2x+1)(e^x)}$

(d) $r(t) = \cos\left(c(\frac{1}{t} - 3c)\right)$, where c is a constant.

chain rule

$r(t) = \cos(\frac{c}{t} - 3c^2)$

$\boxed{r'(t) = (-\sin(\frac{c}{t} - 3c^2)) \cdot (\frac{-c}{t^2})}$

hint: rewrite as ct^{-1}

↳ derivative of $ct^{-1} = -1ct^{-2} = \frac{-c}{t^2}$

2. (9 points) Consider the discrete time dynamical system $x_{t+1} = 2x_t(1 - x_t)$.

(a) (3 points) Find the equilibrium for the DTDS.

$\boxed{\text{2 approaches} \rightarrow \text{both are valid}}$

① $x^* = 2x^*(1-x^*)$ → divide both sides by x^* → need to check if $x^*=0$ is equilibrium

$0 = 2(0)(1-0) \rightarrow 0=0 ✓ \rightarrow \boxed{x^*=0}$

$\frac{x^*}{x^*} = \frac{2x^*(1-x^*)}{x^*} \rightarrow 1 = 2(1-x^*)$

$1 = 2 - 2x^*$

$-1 = -2x^*$

$\boxed{x^* = \frac{1}{2}}$

② $x^* = 2x^*(1-x^*)$

$x^* = 2x^* - 2x^{*2}$

$0 = x^* - 2x^{*2}$

$0 = x^*(1-2x^*)$

split

$\boxed{x^*=0}$

$1 - 2x^* = 0$

$1 = 2x^*$

$\boxed{\frac{1}{2} = x^*}$

(b) (3 points) Is the non-zero equilibrium stable or unstable? Justify your answer using the Stability Theorem.

Stable if $|f'(x^*)| < 1$

$\hookrightarrow f(x_t) = 2x_t(1-x_t)$

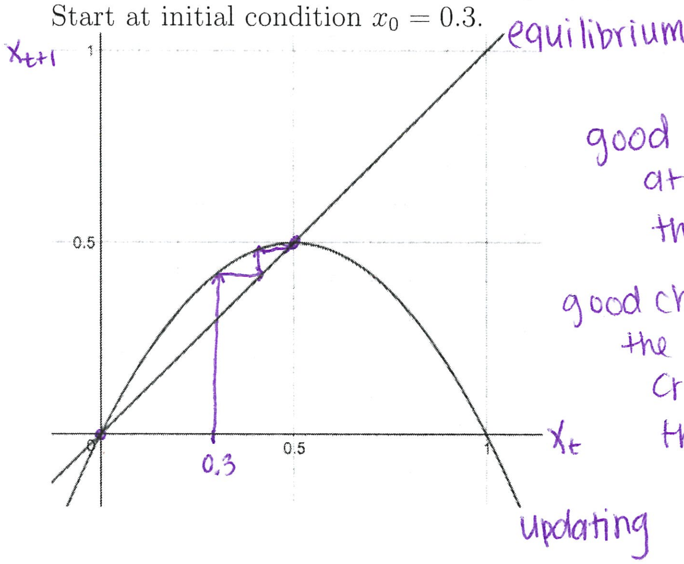
$f(x_t) = 2x_t - 2x_t^2$

$\hookrightarrow f'(x_t) = 2 - 4x_t$

$f'(\tfrac{1}{2}) = 2 - 4(\tfrac{1}{2}) = 2-2 = 0 = f'(\tfrac{1}{2})$

$\hookrightarrow |0| < 1$, so $\boxed{x^* = \tfrac{1}{2} \text{ is stable}}$

(c) (3 points) Check your answer using a graph and cobwebbing. Label your axes. Start at initial condition $x_0 = 0.3$.



good check for part A: both lines cross at 0 and 0.5, so we know that the equilibria we found are correct

good check for part b: the cobweb approaches the equilibrium (point where lines cross) at 0.5, so that also indicates that the equilibrium is stable.

3. (10 points) Consider the logistic growth model.

$$x_{t+1} = \frac{x_t}{r + x_t^2}$$

(a) Assume parameter r is greater than 0. What condition on r guarantees that the equilibrium $x^* = 0$ is stable? Justify your answer with the Stability Theorem. Use correct notation in your work.

Suppose the discrete time dynamical system $X_{t+1} = f(X_t) = \frac{X_t}{r + X_t^2}$ has an equilibrium at $X^* = 0$.

Let $f'(X_t)$ be the derivative of $f(X_t)$ with respect to X_t.

$$f'(X_t) = \frac{1 \cdot (r + X_t^2) - X_t \cdot (2X_t)}{(r + X_t^2)^2} = \frac{r + X_t^2 - 2X_t^2}{(r + X_t^2)^2}$$

$$= \frac{r - X_t^2}{(r + X_t^2)^2}$$

As we know, the equilibrium X^* is stable if $|f'(X^*=0)| < 1$

$$\Rightarrow \left| \frac{r - 0^2}{(r+0)^2} \right| < 1$$

$$\Rightarrow \left| \frac{r}{r^2} \right| < 1$$

$$\Rightarrow \left| \frac{1}{r} \right| < 1$$

r is positive, so $1/r$ is positiv

$0 < \frac{1}{r} < 1$

If $r > 1$ then $1/r < 1$.

so $0 < r < 1$.

(b) Find the positive, non-zero equilibrium in terms of parameter r.

$$X^* = \frac{X^*}{r + (X^*)^2}$$

$$\Rightarrow (r + (X^*)^2) X^* = X^*$$

We can divide by X^* both side.

Assume $X^* \neq 0$ because we are finding non-zero X^*

So we have $(r + (X^*)^2) = 1$

$$\Rightarrow r + (X^*)^2 = 1$$

$$\Rightarrow (X^*)^2 = 1 - r$$

Since we are going to find positive equilibrium.

$$X^* = \sqrt{1 - r}$$

4. (9 points)

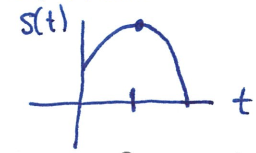
A ball is thrown vertically upward from the roof of a 3 foot building with a velocity of 2 feet per second. The ball's height above the ground after t seconds is given by the function

$$s(t) = 3 + 2t - 16t^2$$

(a) What is the maximum height the ball reaches? Find and justify your answer using properties of the derivative of the function. Check your answer with a graphing calculator.

$S(t) = 3 + 2t - 16t^2$

We should check that at $t = 1/16$ sec the ball has reached a maximum with graph



Velocity: $s'(t) = 2 - 32\cdot(t)$

Set $s'(t) = 0 \Rightarrow 2 - 32t = 0$

$\Rightarrow t = \frac{1}{16}$

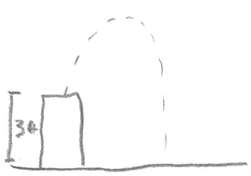
$S(\frac{1}{16}) = 3 + 2\cdot\frac{1}{16} - 16\cdot(\frac{1}{16})^2 = 3 + \frac{2}{16} - \frac{1}{16} = \frac{49}{16}$

(b) For what values of t is the height of the ball increasing? Justify your answer with the derivative.

When the height is increasing, which means the derivative is positive, So $S'(t) > 0$

$\Rightarrow 2 - 32(t) > 0 \Rightarrow t < \frac{1}{16}$ The ball is thrown at $t = 0$, So we have $0 < t < \frac{1}{16}$

(c) What is the velocity of the ball when it hits the ground? The ground has height zero.



$S(t) = 3 + 2t - 16t^2 = 0$

$\Rightarrow 16t^2 - 2t - 3 = 0 \Rightarrow t = \frac{2 \pm \sqrt{(-2)^2 - 4(-3)\cdot 16}}{2\cdot 16}$

$t = \frac{2 \pm \sqrt{4 + 192}}{32} = \frac{2 \pm \sqrt{196}}{32} = \frac{2 \pm 14}{32}$

So $t = \frac{16}{32} = \frac{1}{2} = 0.5$ We don't take the negative value.

Velocity: $S'(t) = 2 - 32t$

When $t = \frac{1}{2}$

$S'(\frac{1}{2}) = 2 - 32\cdot\frac{1}{2} = 2 - 16 = -14$

5. (9 points) Suppose the function $M(t)$ gives the mass of an alien (in grams) and the function V(t) gives the volume of an alien in cubic centimeters. Assume $t > 0$. Let $M(t) = .02e^t$ and $V(t) = 1 + t^2$.

(a) Density is computed by comparing the mass of an object to its volume.

$$Density = \frac{Mass}{Volume}$$

If the mass of an alien increases while the volume of the alien stays constant what happens to the density of the alien?

The density increase.

(b) Write a formula giving the density ($D(t)$) of the alien at time t with mass $M(t)$ and volume $V(t)$.

$$D(t) = \frac{M(t)}{V(t)} = \frac{0.02e^t}{1+t^2}$$

(c) Find the positive value of t where the rate of change of the alien's density is zero. This value of t is also called a critical point.

Take derivative to find critical point.

Quotient rule $\Rightarrow$

$$D'(t) = \frac{0.02e^t \cdot (1+t^2) - 0.02e^t \cdot (2t)}{(1+t^2)^2}$$

$$= \frac{0.02e^t + 0.02e^t \cdot t^2 - 0.02e^t \cdot (2t)}{(1+t^2)^2} = 0$$

$$\Rightarrow 0.02e^t + 0.02e^t \cdot t^2 - 0.02e^t(2t) = 0$$

$$\Rightarrow 0.02e^t(1+t^2-2t) = 0$$

$$\Rightarrow \begin{cases} 0.02e^t = 0 \\ t^2-2t+1=0 \end{cases}$$

$\Rightarrow$ Since we know $e^t \neq 0$. So we can solve $t^2-2t+1=0 \Rightarrow (t-1)^2=0$, $t=1$

Therefore, the critical point is at $t=1$

$$D(1) = \frac{0.02e}{1+1^2} = \frac{0.02e}{2} = 0.01e$$

6. (9 points) Consider the discrete-time dynamical system

$$T_{t+1} = 3(1 - T_t)T_t - hT_t$$

describing a population of fish being harvested at rate h, where $0 \leq h \leq 1$.

(a) Find the nonzero equilibrium population T^* as a function of h.

Equilibrium when $T_{t+1} = T_t$ => Change both to T^*

$T^* = 3(1-T^*)T^* - hT^*$

$T^* = (3 - 3T^*)T^* - hT^*$

$T^* = 3T^* - 3(T^*)^2 - hT^*$

$2T^* - 3(T^*)^2 - hT^* = 0$

$T^*(2-h) - 3(T^*)^2 = 0$

$T^*((2-h) - 3T^*) = 0$

This term gives $T^*=0$.

$2 - h - 3T^* = 0$

$-3T^* = -2 + h$

$\boxed{T^* = \frac{2}{3} - \frac{1}{3}h}$

(b) The equilibrium harvest is given by $P(h) = hT^*$, where T^* is the equilibrium you found in part (a). Find the value of h that maximizes $P(h)$ on the interval $0 \leq h \leq 1$. Use any method to justify answer(including calculator).

$P(h) = hT^*$ => $P(h) = h\left(\frac{2}{3} - \frac{1}{3}h\right)$

$P(h) = \frac{2}{3}h - \frac{1}{3}h^2$

* To find Maximum, you must find crit. points using $f'(h)$, then $f''(h)$ at those crit. values of h to determine concavity / min or max.

Crit. Values

$P'(h) = \frac{2}{3} - \frac{2}{3}h$ => $0 = \frac{2}{3} - \frac{2}{3}h$

$-\frac{2}{3} = -\frac{2}{3}h$ so $h = 1$

Find $F''(h)$ to check Concavity

$P''(h) = -\frac{2}{3}$

$P''(1) = -\frac{2}{3}$ is negative, so is Concave Down — thus there is a max at $\boxed{h=1}$

7. (6 points) Evaluate the following limits. Show all of your work. If you use leading behavior, justify your answer by explaining all of your steps. If you use L'Hopital's Rule, justify why it can be applied each time you use it.

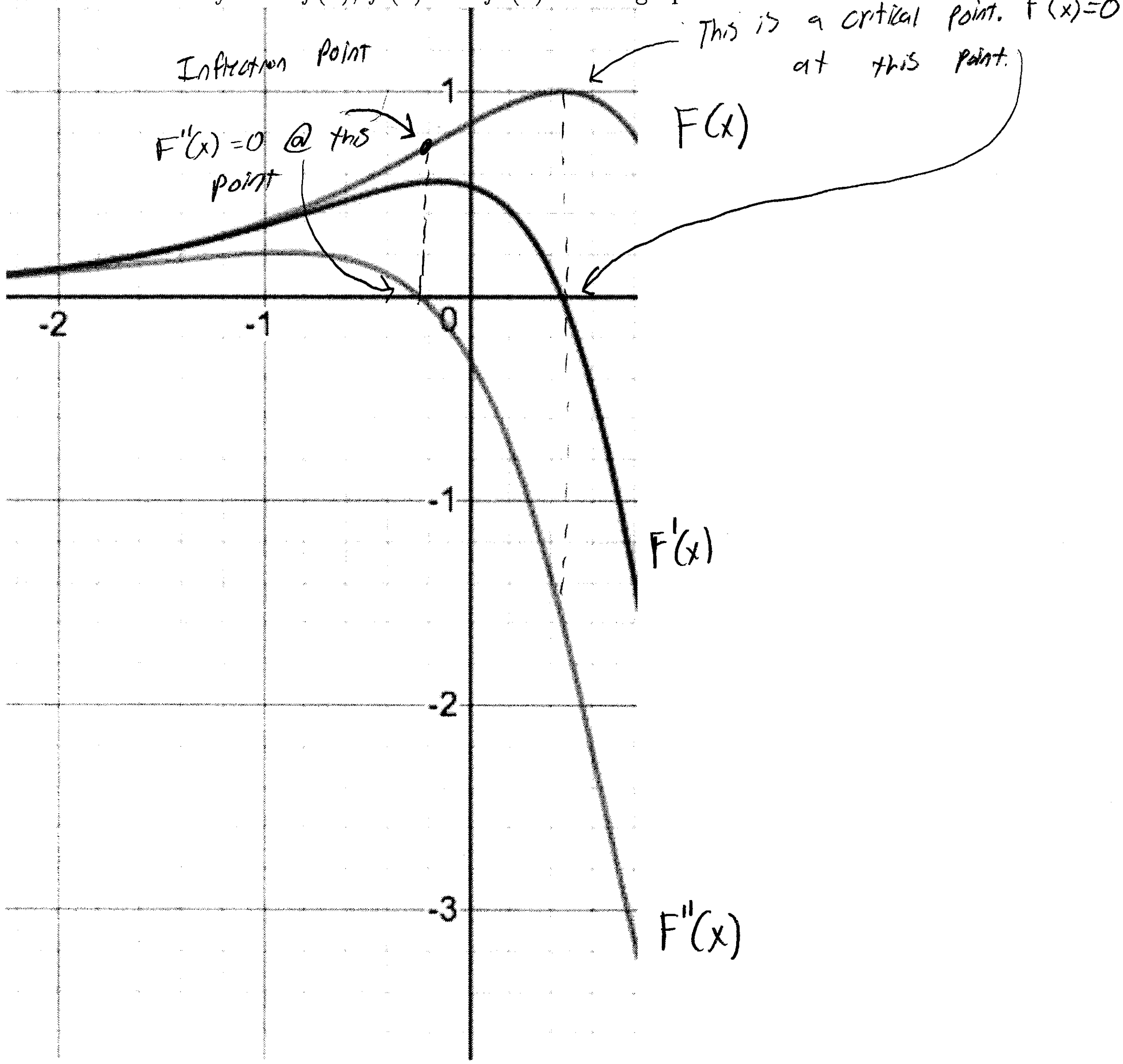
(a) $\lim_{x\to\infty} \frac{5x^2+\frac{1}{x}}{0.05e^x}$

Not on this test (Fall 2018)

(b) $\lim_{x\to 3} \frac{(x-3)}{(x^2-9)}$

(c) $\lim_{x\to\infty} \frac{\ln(x)}{x^2}$

8. (5 points) The following graph shows a function $f(x)$ and its first and second derivatives. Clearly label $f(x)$, $f'(x)$ and $f''(x)$ on the graph.



9. (6 points) Label the given descriptions with the letter of the matching graph. Every graph shows a function $f(x)$.

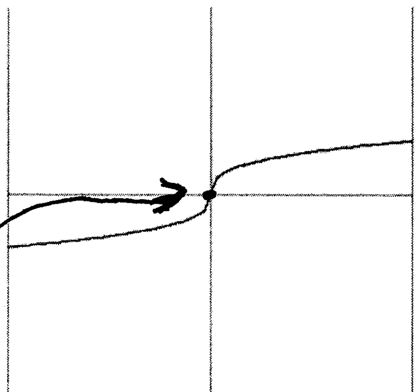
i) $f'(x) = 0$ for two values of x. $f(x)$ changes concavity three times. E

ii) $f'(x)$ is always greater than or equal to 0 and $f''(x) = 0$ at one point. A

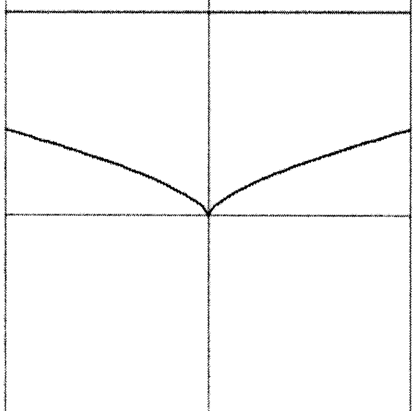
always has positive slope,

point where $f''(x) = 0$

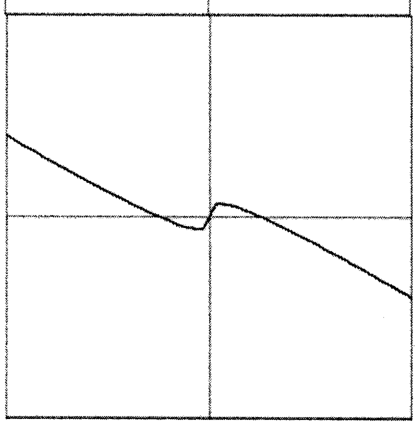
A.



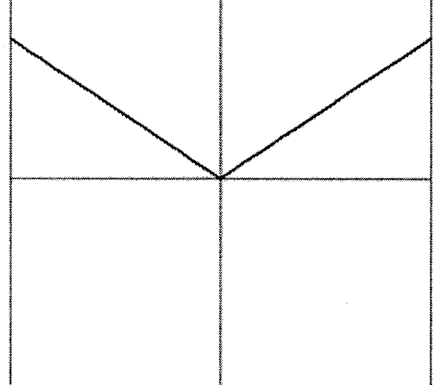
B.



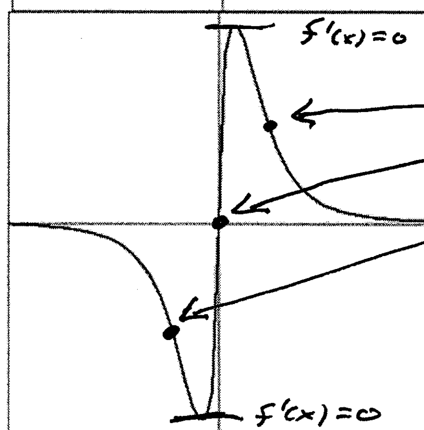
C.



D.

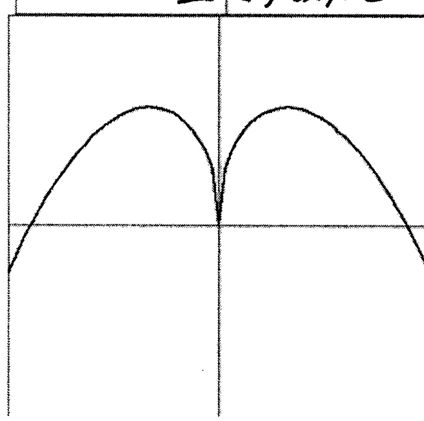


E.



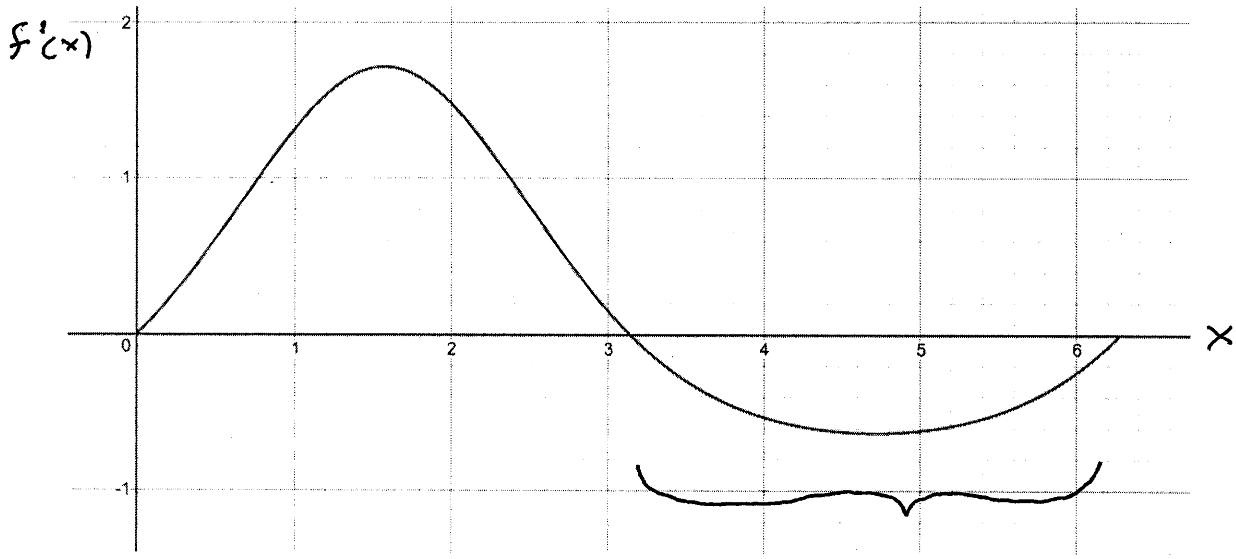
points where concavity changes

F.



For problems 10-13 you can earn 1 point partial credit per problem for correctly labeling vertical axis as either $f(x)$ or $f'(x)$.

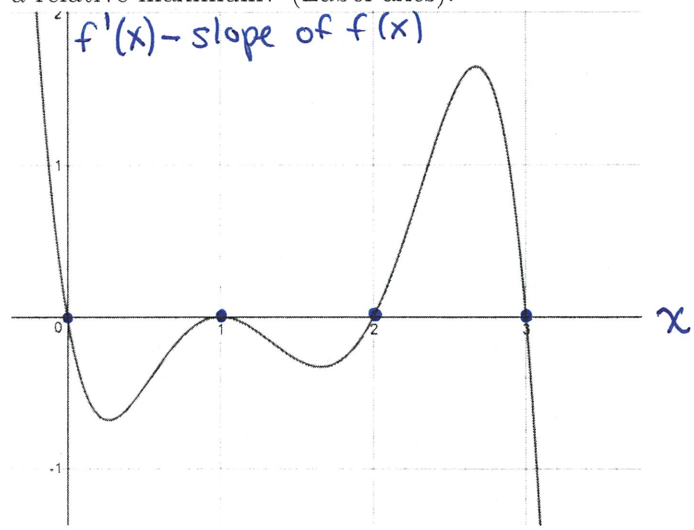
10. (5 points) The following graph shows the *rate of change* $f'(x)$ of a population of fish as a function of time on the closed interval $[0, 6.3]$. What is the largest interval where the population of fish is decreasing?



a) $3.2 < x < 4.6$
b) $1.6 < x < 4.6$
c) $0.6 < x < 2.6$
d) $3.2 < x < 6.3$
e) $2.6 < x < 6.3$

The population is decreasing precisely when the rate of change is negative (when the y-values of the rate of change graph are below the horizontal axis).

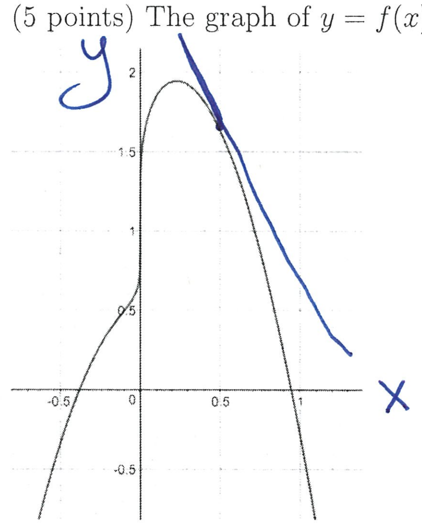
11. The graph of the **derivative** of the function f is shown below. Where does $f(x)$ have a relative maximum? (Label axes).



f(x) has potential relative max when $f'(x)=0$.
when slope of f(x) changes from positive to negative there is a max.

A) $x = 1$ and $x = 2.7$
B) $x = 3$
C) $x = 2$
D) $x = 0$ and $x = 3$
E) $x = 1$

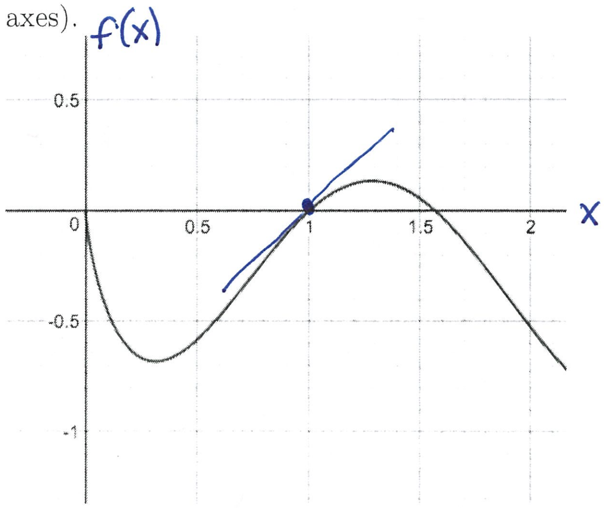
12. (5 points) The graph of $y = f(x)$ is given below. (Label axes).



The value of the **derivative** of f(x) at x = .5 is: the slope of the tangent line at $x=.5$

A) Positive
B) Negative
C) Undefined
D) Zero
E) Not enough information.

13. (5 points) The graph of a function f is shown in the figure below. The function f has a first and a second derivative for all values of x. Which of the following is true? (Label axes).



$f(1)$ – height of $f(x)$ at $x = 1$

$f'(1) =$ slope of $f(x)$ at $x = 1$.

$f''(1) \rightarrow$ related to concavity

A) $f(1) < f'(1) < f''(1)$
B) $f(1) < f''(1) < f'(1)$
C) $f'(1) < f(1) < f''(1)$
D) $f''(1) < f(1) < f'(1)$
E) $f''(1) < f'(1) < f(1)$

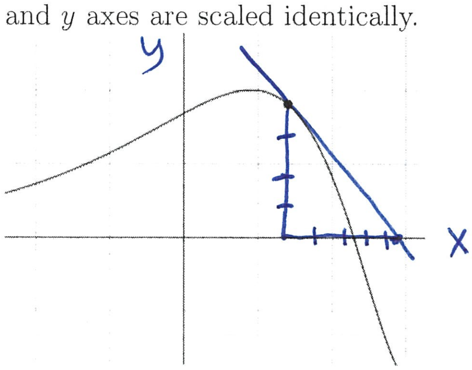
$f(1) = 0$

$f'(1)$ is positive

$f''(1)$ is negative because $f(x)$ is concave down.

$f'' < f < f'$
neg $< 0 <$ pos

14. (5 points) Estimate the slope of the function at the point shown. See below. The x and y axes are scaled identically.



$\frac{\Delta y}{\Delta x}$ means how many times does Δx fit into Δy

A) ~~The slope is approximately $\frac{1}{4}$~~ slope is not positive

B) ~~The slope is approximately $\frac{4}{5}$~~

C) The slope is approximately -20 means $\frac{\Delta y}{\Delta x} = -20$ this means Δy is 20 times as large as Δx.

$\boxed{\text{D)}}$ The slope is approximately $\frac{-4}{5}$

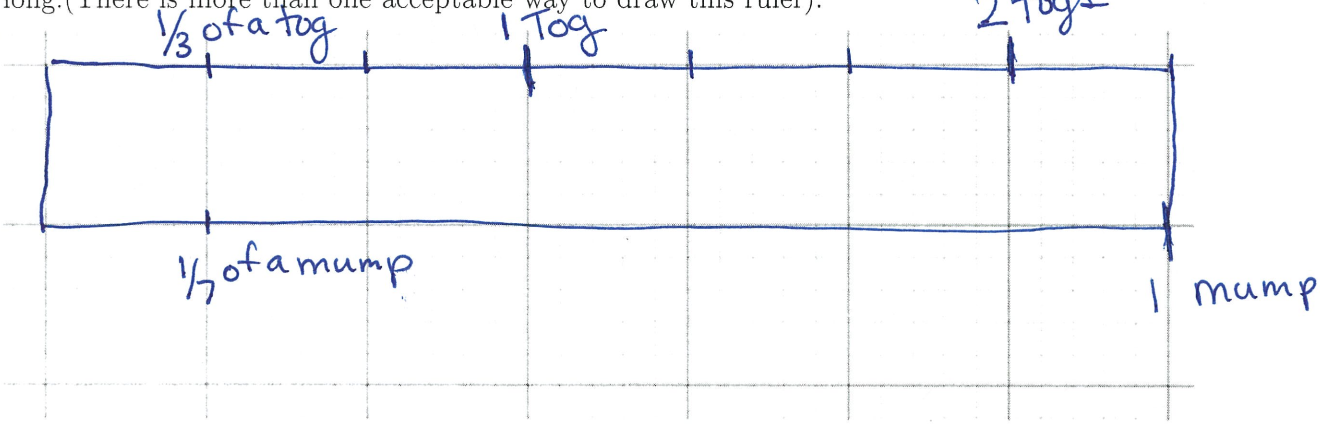
E) The slope is approximately $\frac{-1}{4}$

Bonus-3 points

On an alien planet they use *mumps* and *togs* to measure length.
The length of one mump is $\frac{7}{3}$ times as large as the length of one tog.

1. On the grid paper draw a ruler that has both mumps and togs. The ruler should show that a mump is $\frac{7}{3}$ times as large as a tog. The ruler should be at least 1 mump long.(There is more than one acceptable way to draw this ruler).

1 mump is 7 copies of 1/3 of a tog.



2. Peanut the cat has a length of 33.4 mumps. What is his length in togs?

a) $\frac{(33.4)(7)}{(3)}$ togs

b) $\frac{(33.4)(3)}{(7)}$ togs

Mumps are larger unit of measure as seen on ruler, so more togs fit into Peanut.

3. Garbanzo the cat is measured from nose to tail in both mumps and togs. Garbanzo is Q mumps long and R togs long. Is the measure of Garbanzo's length in mumps (Q) larger or smaller than the measure of his length in togs (R)?

a) His measure in mumps is larger than his measure in togs.

b) His measure in togs is larger than his measure in mumps.

Notice that if you measure Garbanzo with ruler he will have more togs that fit into his length.