Math 155 **Final Exam: Version A** **Spring 2018**

NAME: Key

SECTION: _____ TIME: _____

INSTRUCTOR: _____

Instructions: The exam is closed book and closed notes. You may use an approved calculator, but be sure to show your work on each problem for full credit. Work that is crossed out or erased will not be graded. You can ask for scratch paper from a proctor. Turn in any scratch paper that you use during the exam. You will have two hours to work on the exam.

Problem	Points	Score
1	17	
2	11	
3	21	
4	17	
5	8	
6	16	
7	14	
8	10	
MC1	6	
MC2	6	
MC3	6	
MC4	6	
MC5	6	
MC6	6	
Bonus1	3	
Bonus2	3	
Bonus3	3	
Total	150	

CONFIDENTIALITY PLEDGE

I agree that I will not share any information, either specific or general, about the problems on this examination with any other person until the exams have been returned to us. You can pick up final exams after grades are posted in the front office of Weber, the math building.

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(Signature)

1. (17 points) Suppose that the population w_t of birds satisfies the discrete-time dynamical system

$$w_{t+1} = \frac{2}{7} w_t (k - w_t),$$

where $k > 0$ is a positive parameter.

(a) Find all equilibria.

To find all equilibria, we set up $w^* = \frac{2}{7} w^* (k - w^*)$

First. if $w^* \neq 0$, we can use w^* divide both side, we get

$1 = \frac{2}{7}(k - w^*) \Rightarrow \frac{7}{2} = k - w^* \Rightarrow \underline{w^* = k - \frac{7}{2}}$

If $w^* = 0$, we get $0 = \frac{2}{7} \cdot 0 \cdot (k - 0) = 0$ which left side equal right side.

Therefore, there are two equilibria, they are $w^* = k - \frac{7}{2}$ or $w^* = 0$

(b) For the equilibrium $w^* = 0$, use the Stability Theorem/Criterion to determine the values of k for which that equilibrium is stable. Use correct notation in your steps that show how you used the Stability Theorem/Criterion.

To find out whether the equilibrium is stable, we apply Stability Theorem, which we set up $f(w^*) = \frac{2}{7} w^* (k - w^*)$

And if the equilibrium is stable, $\underline{|f'(w^*)| < 1}$

$f'(w^*) = \frac{2}{7}(1 \cdot (k - w^*) + (-1) \cdot w^*) = \frac{2}{7}(k - w^*) - \frac{2}{7} w^*$

Since $w^* = 0$, $f'(0) = \frac{2}{7} \cdot (k - 0) - \frac{2}{7} \cdot 0 = \frac{2}{7} k$.

which we have $|f'(0)| < 1 \Rightarrow |\frac{2}{7} k| < 1 \Rightarrow -1 < \frac{2}{7} k < 1$

$\Rightarrow \underline{-\frac{7}{2} < k < \frac{7}{2}}$

~~The~~ Hence, when $-\frac{7}{2} < k < \frac{7}{2}$ will make the equilibrium stable.

when $x = \frac{1}{2}$, $f(\frac{1}{2}) = \frac{1}{2} e^{-2 \cdot \frac{1}{4}} = \frac{1}{2} e^{-1/2}$

$x = -\frac{1}{2}$, $f(-\frac{1}{2}) = -\frac{1}{2} e^{-2 \cdot (-\frac{1}{2})^2} = -\frac{1}{2} e^{-1/2}$

Therefore, there are two critical point, $(\frac{1}{2}, \frac{1}{2} e^{-\frac{1}{2}})$, $(-\frac{1}{2}, -\frac{1}{2} e^{-\frac{1}{2}})$.

2. (11 points)

Consider the function $f(x) = xe^{-2x^2}$.

Find *all* values of x where the rate of change of $f(x)$ is zero or undefined. These x values are called critical points of $f(x)$.

First, we will take the derivative to find out the rate of change of $f(x)$. So $f'(x) = e^{-2x^2} + x \cdot e^{-2x^2} \cdot (-4x) = \underline{e^{-2x^2} - 4x^2 e^{-2x^2}}$

Next, we are trying to find critical point which set up $f'(x) = 0$

$\Rightarrow e^{-2x^2} - 4x^2 \cdot e^{-2x^2} = 0 \Rightarrow e^{-2x^2}(1 - 4x^2) = 0 \Rightarrow e^{-2x^2} = 0$ or $1 - 4x^2 = 0$

For $e^{-2x^2} = 0$ we know the expoential function will approach to 0 but never equal to 0. Hence, we can only solve $1 - 4x^2 = 0 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \frac{1}{2}$ or $x = -\frac{1}{2}$.

3. (21 points) Evaluate the following definite and indefinite integrals. Show all of your work.

(a) $\int_1^3 x^{\frac{1}{2}} dx$

$= \frac{2}{3} x^{\frac{3}{2}} \Big|_1^3 = \frac{2}{3} \cdot 3^{\frac{3}{2}} - \frac{2}{3} \cdot 1^{3/2} = 3.464 - \frac{2}{3}$

≈ 2.797

(b) $\int (x \sin(x^2 + 1) + e^2) dx$

$= \int x \sin(x^2+1)\, dx + \int e^2 dx$

$= -\frac{1}{2} \cos(x^2+1) + e^2 x + C$

(c) Use integration by parts to evaluate $\int 5xe^{2x}\, dx$.

By using integration by parts, we can set

$u = x$ $\quad v = \frac{1}{2} e^{2x}$

$du = 1$ $\quad dv = e^{2x}$

$\int 5xe^{2x} dx = 5 \int x e^{2x} dx = 5 \cdot (x \times \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} dx)$

$= 5 \cdot (\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x}) = \frac{5}{2} x e^{2x} - \frac{5}{4} e^{2x} + C$

4. (17 points) $P(t)$ is the position (in meters) of a car at time t (in seconds). The car's velocity is given by

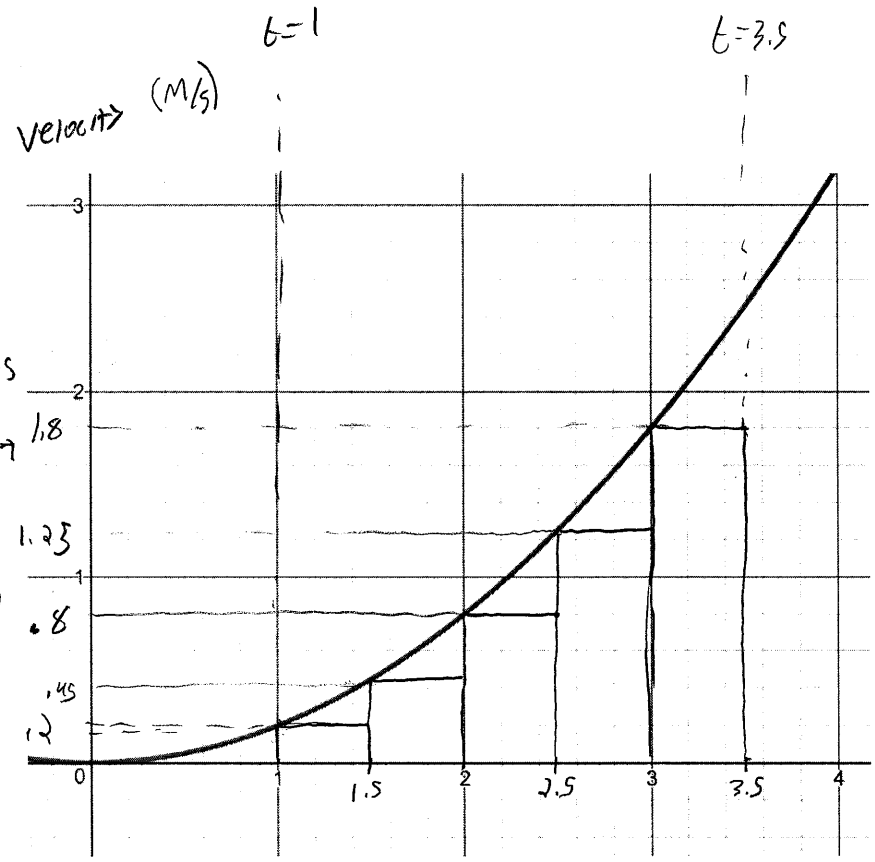
$$\frac{dP}{dt} = 0.2t^2.$$

a) Label both axes on the graph, below, of $y = \frac{dP}{dt}$.

b) Estimate the displacement of $P(t)$ between times $t = 1$ and $t = 3.5$ using a left-hand Riemann Sum with $\Delta t = 0.5$. Draw your rectangles or step functions on the graph below.

$P'(t) = .2t^2$

* plug in t values to find $\frac{dP}{dt}$ values



(b)

(sum of) areas

$P(t) = \sum$ Areas

$P(t) = (.5)(0.2(1)^2) + (.5)(0.2(1.5)^2) + (.5)(0.2(2)^2) + (.5)(0.2(2.5)^2) + (.5)(0.2(3)^2)$

$= (.5)(.2) + (.5)(.45) + (.5)(.8) + (.5)(1.25) + (.5)(1.8)$

$\boxed{= 2.25 \text{ meters}}$

c) Find the exact value of the displacement in car's position $P(t)$ between $t = 1$ and $t = 3.5$ using integration.

$$P(t) = \int_1^{3.5} .2t^2\,dt$$

Anti-der value

$$\frac{.2t^{2+1}}{2+1} = \frac{.2}{3}t^3 = \frac{1}{15}t^3$$

So...

$$P(t) = \frac{1}{15}t^3 \Big|_1^{3.5} = \frac{1}{15}(3.5)^3 - \frac{1}{15}(1)^3 = 2.858 - .0666$$

* evaluated from $t = 1$ to $t = 3.5$

$\boxed{= 2.791 \text{ meters}}$

* This makes sense because our Riemann Sum approx will be an underestimate so the real answer will be greater.

5. (8 points) Peanut the cat starts with a concentration of medicine in his bloodstream equal to 5 milligrams per liter (mg/L). Each morning the cat has used up 13% of the medicine in his bloodstream. Each afternoon the cat gets enough medication to increase the concentration of medicine in his bloodstream by 3.5 mg/L. Let M_t = concentration of medicine on day t.

Write down a discrete-time dynamical system, together with an initial condition, that describes this situation.

$$M_{t+1} = .87 M_t + 3.5 \quad , \quad M_0 = 5 \ \frac{mg}{L}$$

6. (16 points) Let $P(t)$ equal the position of a toy car at time t (in seconds). Suppose that

$$\frac{dP}{dt} = 0.7\sin(4t + \pi).$$

(a) Use a definite integral to determine the displacement in the position of the car between times $t = \frac{\pi}{4}$ and $t = 3\pi$. Show work. You will only get partial credit for finding the final answer with your calculator if there is not work.

$$P(t) = \int_{\frac{\pi}{4}}^{3\pi} .7 \sin(4t + \pi)\, dt$$

$$\rightarrow = .7 \int_{\frac{\pi}{4}}^{3\pi} \sin(4t+\pi)\, dt$$

$$= .7 \left[-\frac{1}{4}\cos(4t+\pi) \right]_{\frac{\pi}{4}}^{3\pi}$$

$$= .7 \left(-\frac{1}{4}\cos(4(3\pi)+\pi) - \left(-\frac{1}{4}\cos(4(\tfrac{\pi}{4})+\pi)\right)\right)$$

$$= .7 \left(-\frac{1}{4}\cos(12\pi+\pi) - \left(-\frac{1}{4}\cos(\pi+\pi)\right)\right)$$

$$= .7 \left(-\frac{1}{4}\cos(13\pi) + \frac{1}{4}\cos(2\pi)\right)$$

$$= .7 \left(\frac{-1}{4}(-1) + \frac{1}{4}(1)\right)$$

$$= .7 \left(\frac{1}{4} + \frac{1}{4}\right) \boxed{= .35 \text{ units}}$$

include units if specified.

(b) Determine $P(t)$ if $P(0) = 7$. (That is, find a solution to the differential equation $\frac{dP}{dt} = 0.7\sin(4t + \pi)$ with initial condition $P(0) = 7$.)

$$\frac{dP}{dt} = .7 \sin(4t+\pi)$$

$$\rightarrow dP = .7 \sin(4t+\pi)\, dt$$

$$\rightarrow \int dP = \int .7 \sin(4t+\pi)\, dt$$

$$P(t) = \int .7 \sin(4t+\pi)\, dt$$

$$= .7 \int \sin(4t+\pi)\, dt$$

$$= .7 \left[-\frac{1}{4}\cos(4t+\pi) \right] + C$$

$$= \frac{7}{10}\left(-\frac{1}{4}\right)\cos(4t+\pi) + C$$

$$P(t) = -\frac{7}{40}\cos(4t+\pi) + C$$

* Plug in initial cond.

$$7 = -\frac{7}{40}\cos(4(0)+\pi) + C$$

$$7 = -\frac{7}{40}\cos(\pi) + C$$

$$7 = -\frac{7}{40}(-1) + C$$

$$7 = \frac{7}{40} + C$$

$$C = 7 \cdot \frac{40}{7} = 40$$

$$\boxed{P(t) = -\frac{7}{40}\cos(4t+\pi) + 40}$$

7. (14 points) A population p_t of mice obeys the discrete-time dynamical system

$$p_{t+1} = 1.4p_t.$$

(a) Write down the solution to this discrete-time dynamical system if $p_0 = 642$. (In other words, find a function that gives population for any time t.)

$$p_t = 642 \times 1.4^t$$

(b) If $p_0 = 642$, at what time will the population reach size 2000?

$$\frac{2000}{642} = \frac{642 \times 1.4^t}{642} \rightarrow \frac{2000}{642} = 1.4^t$$

$$\ln\left(\frac{2000}{642}\right) = \ln(1.4^t) \rightarrow \ln\left(\frac{2000}{642}\right) = t \times \ln(1.4)$$

t rolls down using log-exponent rule

$$t = \frac{\ln\left(\frac{2000}{642}\right)}{\ln(1.4)} \rightarrow \boxed{t = 9.2586 \text{ [units]}}$$

8. (10 points) Peanut the cat gained too much weight and needs to go on a diet. Peanut's veterinarian says that Peanut should eat 200 calories per day. Taste of the Wild cat food has 27 calories per cubic inch of food. You want to measure Peanut's food with your US measuring cups. How many cups of food should Peanut get each day? You know the following unit conversions:

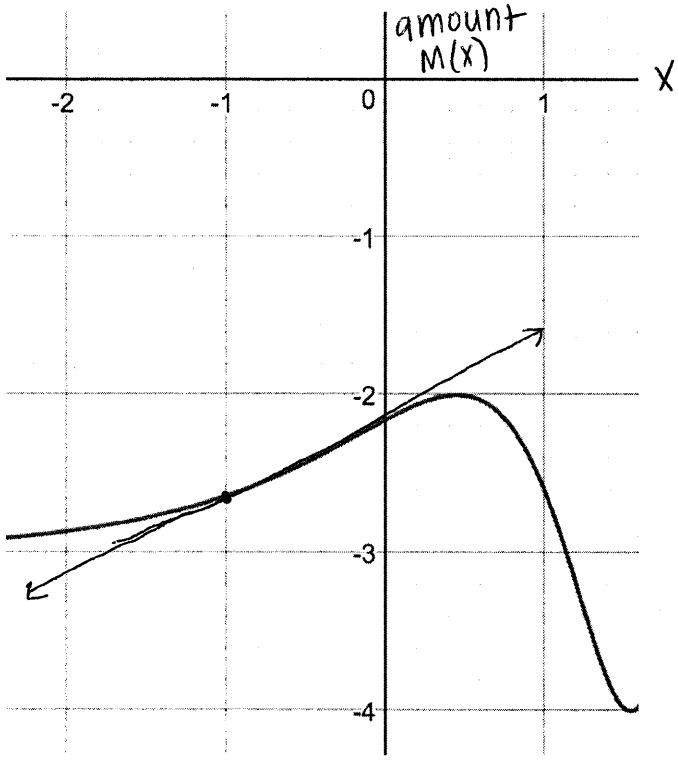
1 US cup = 236.46 cubic centimeters
1 inch is 2.54 times as large as 1 centimeter

$$\frac{200\text{ cal}}{\text{day}} \times \frac{1\text{ in}^3}{27\text{ cal}} \times \frac{(2.54)^3\text{ cm}^3}{1^3\text{ in}^3} \times \frac{\text{US cup}}{236.46\text{ cm}^3} = \boxed{0.5133 \frac{\text{US cups}}{\text{day}}}$$

½ cup of food is reasonable. Answers to real-world problems should be logical because test writer looks up actual quantities.

START OF MULTIPLE CHOICE. **Please Bubble Version A on Scantron.** Please bubble answers on scantron form starting at problem 1 on scantron. Scantron must have your student ID to be counted for a grade. Each multiple choice problem is worth 6 points.

MC 1. The graph of the **mass** of an object $y = M(x)$ is given below.



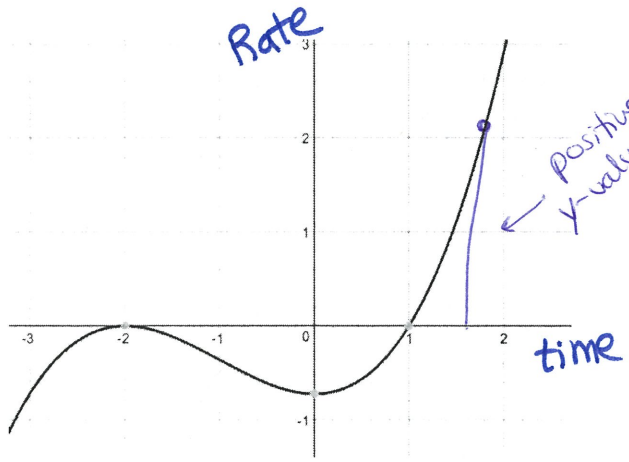
The value of the **rate of change** of mass at time $x = -1$ is:

→ slope of tangent line at $x = -1$

A) Positive
B) Negative
C) Undefined
D) Zero
E) Not enough information.

The slope is positive.

MC 2. The graph below shows the **rate of change** of the mass of a bacterial culture. Where is the **mass** of the bacterial culture increasing on the interval $-3 < x < 2$?

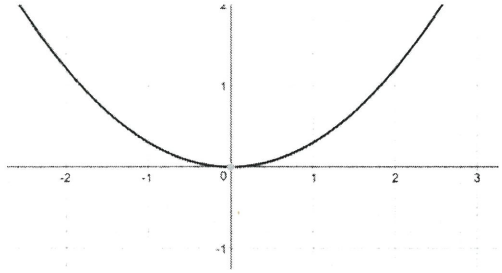


A) I. $-3 < x < -2$
B) II. $0 < x < 1$
C) III. $1 < x < 2$
D) All three intervals: I, II, and III
E) Both I and II

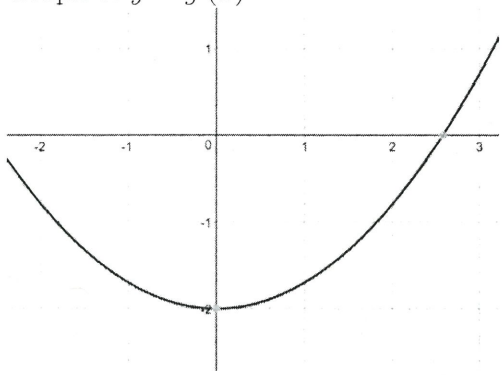
The mass increases anywhere the rate of change of the mass is positive.

The y-values of the rate of change graph are the slopes of the mass versus time graph.

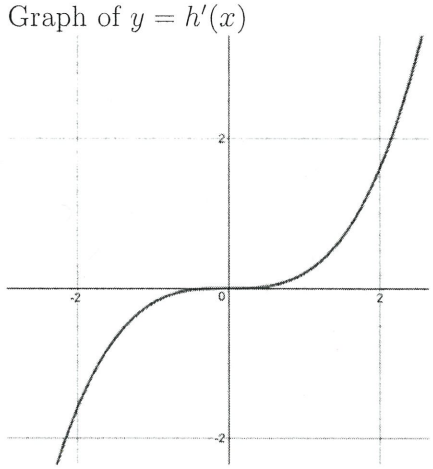
MC 3. Graph of $y = f'(x)$



Graph of $y = g'(x)$



Graph of $y = h'(x)$



Which of the following is true?

A) $\int_{-2}^{2} f'(x)dx < \int_{-2}^{2} g'(x)dx < \int_{-2}^{2} h'(x)dx$

B) $\int_{-2}^{2} g'(x)dx < \int_{-2}^{2} h'(x) < \int_{-2}^{2} f'(x)dx$

C) $\int_{-2}^{2} h'(x)dx < \int_{-2}^{2} f'(x)dx < \int_{-2}^{2} g'(x)dx$

D) $\int_{-2}^{2} f'(x)dx < \int_{-2}^{2} h'(x)dx < \int_{-2}^{2} g'(x)dx$

$\int_{-2}^{2} f'(x)dx =$ area between rate function and x-axis.

$\int_{-2}^{2} f'(x)dx$ means if $f'(x)$ is rate of change, the definite integral tells us how much $f(x)$ changed from $x=-2$ to $x=2$.

$\int_{-2}^{2} f'(x)dx$ is positive

$\int_{-2}^{2} g'(x)dx$ is negative

$\int_{-2}^{2} h'(x)dx$ is zero

MC 4. A centimeter is 10 times as long as a millimeter. We measured Mango the cat and found out she is x millimeters long. If we measure her in centimeters how long will she be?

A) $10x$ centimeters.

B) $\frac{1}{10}x$ centimeters.

C) $10x$ millimeters.

D) $\frac{1}{10}x$ millimeters.

E) None of the above.

Conversion factor: $\frac{10 \text{ mm}}{1 \text{ cm}}$

$$x \text{ cm} \times \frac{10 \text{ mm}}{1 \text{ cm}} = \boxed{10x \text{ mm}}$$

Because mm are smaller than cm, we know the length of Mango is measured by a <u>larger</u> amount of mm than cm.

MC 5.
How many cubic centimeters are equal to 716 cubic inches?

(2.54 cm = 1 inch)

A) (716)(2.54) cubic centimeters

B) $(716)(2.54^3)$ cubic centimeters

C) $\frac{716}{2.54}$ cubic centimeters

D) $\frac{716}{2.54^3}$ cubic centimeters

E) $\frac{2.54^3}{716}$ cubic centimeters

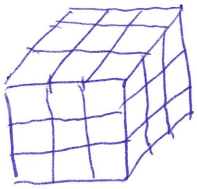
$$716 \text{ in}^3 \times \left(\frac{2.54 \text{ cm}}{1 \text{ in}}\right)^3 = \boxed{(716)(2.54^3) \text{ cm}^3}$$

Note:

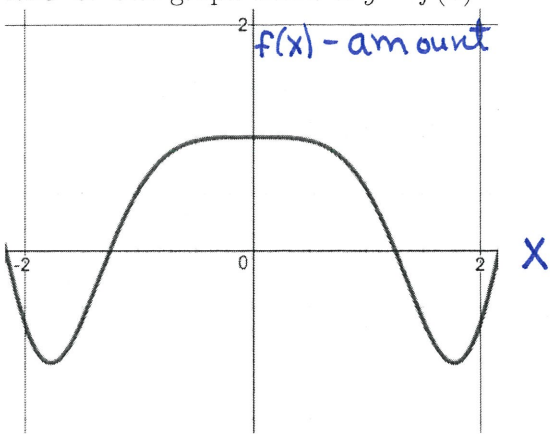
$$2.54 \text{ cm}^3 \neq 1 \text{ in}^3$$

$$\boxed{2.54^3 \text{ cm}^3 = 1 \text{ in}^3}$$

There are lots of cubic cm in 1 cubic inch →



MC 6. The graph below is $y = f(x)$

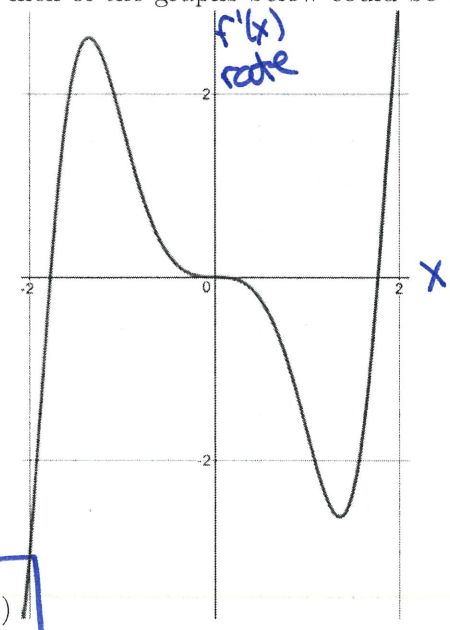


at $x=0$, $f(x)$ has relative max

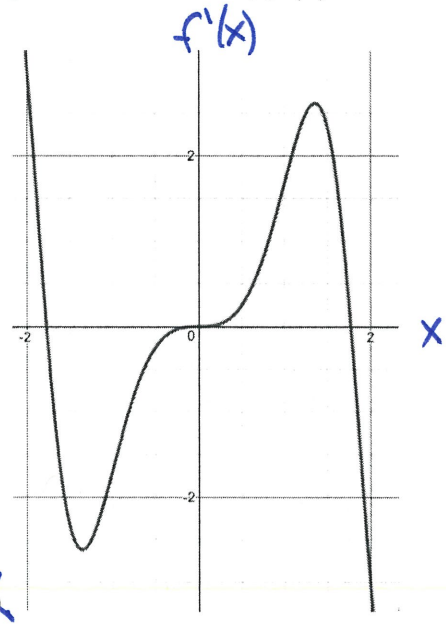
so $f'(x) \geq 0$ right before $x=0$

and $f'(x) \leq 0$ right after $x=0$.

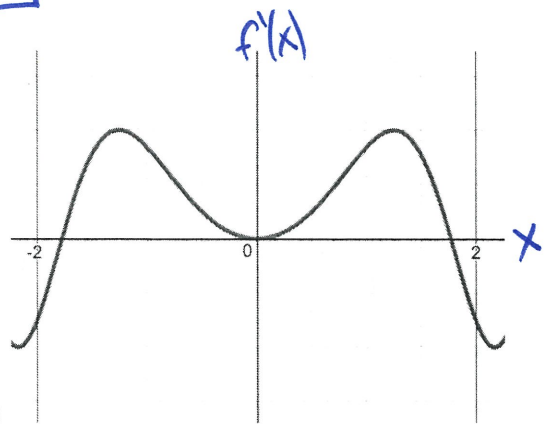
Which of the graphs below could be the graph of its derivative, $f'(x)$?



A)



~~B)~~

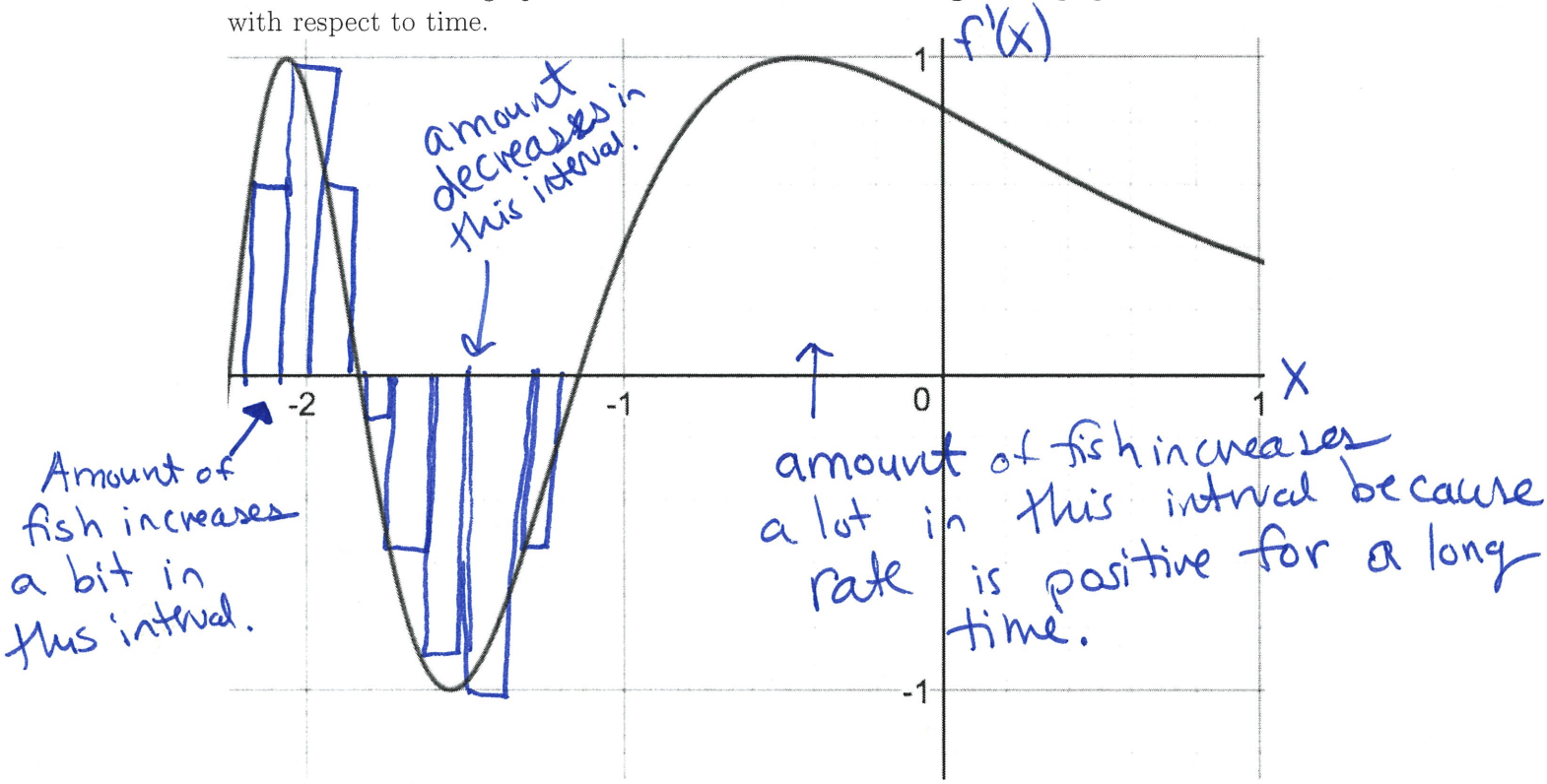


C)

Graph C shows $f'(x)$ is positive from $x = -1.75$ to $x = 1.75$.

$f(x)$ does not have a positive slope on that entire interval.

MC7 Bonus. The graph below shows the **rate of change of a population** of fish with respect to time.



Where on the interval from $-2 \leq t \leq 1$ is the **population** the largest? In other words, where does population have a **global maximum**? (Label axes).

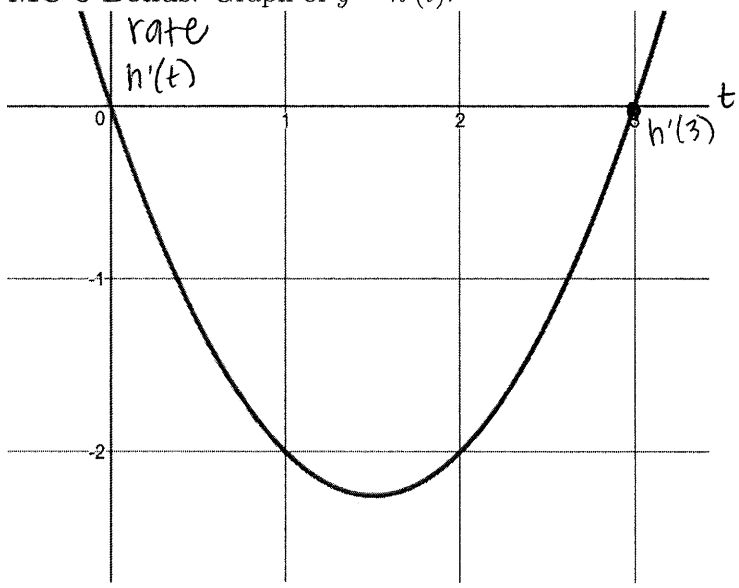
A) $t = -2$
B) $t = -1.8$ Relative max, but not global max.
C) $t = -1.15$
D) $t = 1.6$
$\boxed{\text{E) } t = 1}$

Relative max can occur when $f'(x)=0$, when $f'(x)$ is undefined or ~~when~~ at endpoints when $f(x)$ has relative max $f'(x)$ changes from positive y-values to negative y-values.

$t=-1.8$ and $t=1$ are 2 relative max.

The global max is at $t=1$. Use Riemann sums to approximate total accumulated change in fish. The area between rate curve and x-axis gives Δ amount of fish.

MC 8 Bonus. Graph of $y = h'(t)$.



graph of $h'(t)$ is given

$h'(3) = 0$ → found by looking at the "y" value

$h''(3) > 0$ → $h''(t)$ is the derivative (slope of the tangent line) of $h'(t)$. At $t=3$, the tangent line has a positive slope.

The graph of a function $h'(t)$ is shown above. Let

$$h(x) = \int_0^x h'(t)dt$$

→ $h(3) < 0$ → $h(3)$ is the area under the curve from 0 to 3. The area is negative, so $h(3)$ is less than zero.

Which of the following is true?

A) $h(3) < h'(3) < h''(3)$

B) $h(3) < h''(3) < h'(3)$

C) $h''(3) < h'(3) < h(3)$

D) $h''(3) < h(3) < h'(3)$

E) $h'(3) < h(3) < h''(3)$

$h(3) < h'(3) < h''(3)$

MC 9. Bonus

The table below gives information about functions f and g. Let h be defined as $h(x) = f(g(x))$. What is the rate of change of h at $x = 4$?

$x =$	1	2	3	4
$f(x)$	20	23	18	14
Rate of change of f at x	7	-3	-7	-2
$g(x)$	-10	-11	-4	2
Rate of change of g at x	-0.5	2	4	3

a) -2 b) -3 c) -6 d) -7 e) -9

chain rule

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(4) = f'(g(4)) \cdot g'(4)$

$h'(4) = f'(2) \cdot 3$

$h'(4) = (-3) \cdot 3$

$\boxed{h'(4) = -9}$